1.4.13

HW 5 Solutions

Solve the following differential equations:

(a) $x\frac{dy}{dx} - ky = x^2 \quad (k \quad \text{constant})$,

(b) $\frac{dy}{dx} - y\tan x = x$,

(c) $\frac{dy}{dx} + y\tan x = \sec x$,

(d) $\frac{dy}{dx} + (y - 2\sin x)\cos x = 0$,

(e) $(x^2-1)\frac{dy}{dx} + 2y = (x+1)^2$,

(f) $x^2 \log x dy + (xy-1)dx = 0$,

(g) $2xy\frac{dy}{dx} - y^2 = x^2$,

(h) $ydx = (x+y^2)dy$,

Solution

Using Formulas:

$$\frac{dy}{dx} + a(x)y = h(x)$$

$$p = \exp(\int a(x)dx)$$

$$y = \frac{1}{p}\int phdx + \frac{c}{p}$$

Equation	**a(x)**	**x**	**p(x)**	**y(x)**
(a)				
$xy' - ky = x^2$	$\frac{-k}{x}$	x	x	$y = \frac{x^2}{2-k} + cx^k, \quad k \neq 2$
$xy' - 2y = x^2$	$\frac{-2}{x}$	x	x^{-2}	$x^2\ln\lvert x\rvert + cx^2, \quad k = 2$
(b)				
$y' - y\tan(x) = x$	$-\tan(x)$	x	$\cos(x)$	$y(x) = x\tan(x) + 1 + \cancel{\csc(x)}$ $c\sec(x)$
(c)				
$y' + y\tan(x) = \sec(x)$	$\tan(x)$	$\sec(x)$	$\sec(x)$	$y(x) = c\cos(x) + \sin(x)$
(d)				
$y' + \cos(x)y = 2\cos(x)\sin(x)$	$\cos(x)$	$2\cos(x)\sin(x)$	$e^{\sin(x)}$	$y(x) = 2\sin(x) - 2 + \frac{c}{e^{\sin(x)}}$
(e)				
$y' + \frac{2y}{x^2-1} = \frac{(1+x)^2}{x^2-1}$	$\frac{2}{x^2-1}$	$\frac{(1+x)^2}{x^2-1}$	$\frac{x-1}{x+1}$	$y(x) = \frac{(x+1)(c+x)}{x-1}$

Equation	a(x)	x	p(x)	y(x)
(f) $y' + \frac{y}{x\log(x)} = \frac{1}{x^2\log(x)}$	$\frac{1}{x\log(x)}$	$\frac{1}{x^2\log(x)}$	$\log(x)$	$y(x) = \frac{cx-1}{x\log(x)}$
(g) solve for $w = y^2$ $w'(x) - \frac{w(x)}{x} = x$	$\frac{-1}{x}$	x	$\frac{1}{x}$	$w(x) = x(c+x)$ $y^2(x) = x(c+x)$
(h) solve for $y = x(y)$ $x'(y) - \frac{x(y)}{y} = y$	$\frac{-1}{y}$	y	$\frac{1}{y}$	$x(y) = y(c+y)$

1.5.18

Solve the following differential equations:

(a) $\frac{d^4y}{dx^4} - k^4y = 0$ (vibration of a beam,)

(b) $\frac{d^2y}{dx^2} + y = \sin x$,

(c) $\frac{d^2y}{dx^2} - y = \sin x$,

(d) $\frac{d^2y}{dx^2} - y = e^x$,

(e) $\frac{d^2y}{dx^2} - y = xe^x$,

(f) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \sin x$,

(a) $r^2 - r - 2 = 0$

$(r+1)(r-2) = 0, \quad r_k \in \{-1, 2\}, \quad y = C_1e^{-x} + C_2e^{2x}$

(b) $r^3 - r^2 - r + 1 = 0$

$(r-1)^2(r+1) = 0, \quad r_k \in \{1, 1, -1\}, \quad y = C_1e^{-x} + (C_2 + C_3x)e^x$

cos(x)

(c) $r^2 - 2r + 2 = 0, \quad r_k \in \{1+i, 1-i\}, \quad y = e^x(C_1\cancel{\csc}(x) + C_2\sin(x))$

(d) $r^4 - 4r^3 + 7r^2 - 6r + 2 = 0, \quad r_k \in \{1, 1, 1+i, 1-i\}, \quad y = e^x(C_1 + C_2 \overset{x}{\wedge} + C_3\cos(x) + C_4\sin(x))$

(e) $r^3 - 1 = 0, \quad r_k \in \left\{1, \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \frac{-1}{2} - i\frac{\sqrt{2}}{2}\right\}, \quad y = C_1 e^x + e^{-x/2}\left[C_2 \cos(\frac{\sqrt{3}x}{2}) + C_3 \sin(\frac{\sqrt{3}x}{2})\right]$

(f) $r^2 - 2i = 0, \quad r_k \in \{-1-i, 1+i\}, \quad y = C_1 e^{-(1+i)x} + C_2 e^{(1+i)x}$

1.5.20

Use the method of undetermined coefficients to find the complete solution of each of the following differential uations:

(a) $\frac{d^2y}{dx^2} + k^2 y = \sin x, \quad (k^2 \neq 0, 1)$

(b) $\frac{d^2y}{dx^2} + y = \sin x$

(c) $\frac{d^2y}{dx^2} - y = \sin x,$

(d) $\frac{d^2y}{dx^2} - y = e^x$

(e) $\frac{d^2y}{dx^2} - y = xe^x,$

(f) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \sin x$

(g) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \sin x$

(h) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + +20y = 4x^2 e^{3x}$

(i) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{-x} + 10\sin x - 4x$

Solution:

(a)

Hom Sol: $A\cos kx + B\sin kx$

RHS Family: $\{\sin x, \cos x\}$

Ansetz: $y_p = a\sin x + b\cos x$

$$y_p'' + k^2 y_p = -a\sin x - b\cos x + k^2 a \sin x + k^2 b \cos x \equiv \sin x$$

$$a(k^2 - 1) = 1, \quad b(k^{2-1}) = 0$$

$$a = \frac{1}{k^2 - 1}$$

$$y = y_h + y_p = A\cos(kx) + B\sin(kx) + \frac{\sin x}{k^2 - 1}, \quad k \neq 0, 1$$

(b)

Hom Sol: $A\sin x + B\cos x$

RHS Family: $\{\sin x, \cos x\} \to \{x\sin(x), x\cos(x)\}$

Ansetz: $y_p = \cancel{a}\sin x + bx\cos x$

$y_p'' + y_p = 2a\cos x - 2b\sin x \equiv \sin x$

$a = 0, \quad b = \frac{1}{2}$

$y = A\sin x + B\cos x - \frac{1}{2}x\cos x$

(c)

Hom Sol: $y_h = Ae^{x} + Be^{-x}$

RHS Family: $\{\sin x, \cos x\}$

Ansetz: $yp = a\sin x + b\cos x$

$y_p'' + y = -2a\sin x - 2b\cos x \equiv \sin x \Rightarrow b = 0, \quad a = \frac{-1}{2}$

$y = Ae^{x} + Be^{-x} - \frac{1}{2}\sin x$

(d)

Hom Sol: $y_h = Ae^{x} + Be^{-x}$

RHS Family: $\{e^{x}\}$, ~~modified~~ RNS Family $\{xe^{x}\}$

Ansetz: $y_p = axe^{x}$

$y_p'' + y_p = a(2e^{x} + xe^{x} - xe^{x}) \equiv e^{x} \Rightarrow A = \frac{1}{2}$

$y = Ae^{x} + Be^{-x} + \frac{1}{2}e^{x}x$

(e)

$y_h = Ae^{x} + Be^{-x}$

RHS Family: $\{xe^{x}, e^{x}\}$, modified RNS Family $\{x^2e^{x}, xe^{x}\}$

Ansetz: $y_p = ax^2e^{x} + bxe^{x}$

$y_p'' + y_p = e^{x}[b(2+x) + a(2+4x+x^2)] - ax^2e^{x} - bxe^{x} \equiv xe^{x}$

$2b + a(2+4x) \equiv x \Rightarrow 4a = 1, \quad a = \frac{1}{4}, \quad b = -a, \quad b = \frac{-1}{4}$

$y = Ae^{x} + Be^{-x} + \frac{1}{4}(x^2 - x)e^{x}$

(f)

$y_h = e^{x}(A\cos x + B\sin x)$

RHS Family: $\{\sin x, \cos x\}$

Ansetz: $y_p = a\sin x + b\cos x$

$y_p'' + 2y_p' + 2y = (-2a+b)\cos x + (a+2b)\sin x \equiv \sin x$

$\Downarrow$

$b = 2a \qquad 5a = 1, \quad a = \frac{1}{5} \quad b = \frac{2}{5}$

$Y = Y_h + Y_p = e^{x}(A\cos x + B\sin x) + \frac{1}{5}(2\cos x + \sin x)$

$$x = c_1 \cos k\omega_1 t + c_2 \sin k\omega_1 t + c_3 \cos k\omega_2 t + c_4 \sin k\omega_2 t$$

$$y = \sqrt{2}\left(-c_1 \cos k\omega_1 t - c_2 \sin k\omega_1 t + c_3 \cos k\omega_2 t + c_4 \sin k\omega_2 t\right)$$

where $\omega_1 = \sqrt{2-\sqrt{2}}$, $\omega_2 = \sqrt{2+\sqrt{2}}$

..8.33

$$x = c_1 \cos k\omega_1 t + c_2 \sin k\omega_1 t + c_3 \cos k\omega_2 t + c_4 \sin k\omega_2 t$$

$$y = \sqrt{2}\left(c_1 \cos k\omega_1 t + c_2 \sin k\omega_1 t - c_3 \cos k\omega_2 t - c_4 \sin k\omega_2 t\right)$$

where $\omega_1 = \sqrt{2+\sqrt{2}}$, $\omega_2 = \sqrt{2-\sqrt{2}}$

this is for D)

D)

$A = 1$, $B = 1$

~~$C = 1$, $D = -1$~~

~~Solution~~ $\cancel{x = e^{2t} + 1}$

$\cancel{y = e^{2t} - 1}$

1.8.33

f) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + \dfrac{dy}{dt} + 3x + y = 0 \Rightarrow (D^2 + D + 3)x + (D+1)y = 0$ — (1a)

$\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + x + y = 0 \Rightarrow (D^2+1)x + (D+1)y = 0$ — (1b)

$$\Delta = \begin{vmatrix} D^2+D+3 & D+1 \\ D^2+1 & D+1 \end{vmatrix} = (D+1)(D+2)$$

Characteristic equation $(r+1)(r+2) = 0$

$\Rightarrow$ $r = -1, -2$

$x_H = c_1 e^{-t} + c_2 e^{-2t}$ — (2)

$y_H = d_1 e^{-t} + d_2 e^{-2t}$ — (3)

$$\Delta x = \begin{vmatrix} 0 & D+1 \\ 0 & D+1 \end{vmatrix} = 0 \Rightarrow x_p = 0, \quad x = x_H$$

$$\Delta y = \begin{vmatrix} D^2+D+3 & 0 \\ D^2+1 & 0 \end{vmatrix} = 0 \Rightarrow y_p = 0, \quad y = y_H$$

$$-\cancel{c_1 e^{-t}} - 2c_2 e^{-2t} + \cancel{c_1 e^{-t}} + 4c_2 e^{-2t} + 3c_1 e^{\ } + 3c_2 e^{\ }$$

$$-\cancel{d_1 e^{-t}} - 2d_2 e^{-2t} + \cancel{d_1 e^{-t}} + d_2 e^{-2t} = 0$$

$$\Rightarrow \quad \underline{c_1 = 0} \qquad -d_2 + 5c_2 = 0 \Rightarrow \underline{d_2 = 5c_2}$$

Substituting (2) and (3) in (1b) gives.

$$-d_1 e^{-t} + 2d_2 e^{-2t} + d_1 e^{-t} + d_2 e^{-2t} + \cancel{c_1 e^{-t}} + 4c_2 e^{-2t}$$

$$+ \cancel{c_1 e^{-t}} + c_2 e^{-2t} = 0 \qquad \underline{d_2 = 5c_2}$$

Solution

$$\boxed{\begin{aligned} x &= c_2 e^{-2t} \\ y &= d_1 e^{-t} + 5c_2 e^{-2t} \end{aligned}}$$

Renaming c_2 as c_1 and d_1 as c_2, we have.

$$\boxed{\begin{aligned} x &= c_1 e^{-2t} \\ y &= 5c_1 e^{-2t} + c_2 e^{-t} \end{aligned}}$$

(g) $\dfrac{dx}{dt} - 2x = 0 \Rightarrow (D-2)x = 0$ — (1a)

$-3x + \dfrac{dy}{dt} + 2y = 0 \Rightarrow \cancel{(-3+D)}\ -3x + (D+2)y = 0$ — (1b)

$-2y + \dfrac{dz}{dt} - 3z = 0 \qquad -2y + (D-3)z = 0$ — (1c)

4.2.5) a) $\frac{d^2y}{dx^2} = xy \Rightarrow \frac{d^2y}{dx^2} - xy = 0$

$$\sum_{k=0}^{\infty} k(k-1)A_k x^{k-2} - x\sum_{k=0}^{\infty} A_k x^k = 0$$

$$\sum_{k=0}^{\infty} k(k-1)A_k x^{k-2} - \sum_{k=0}^{\infty} A_k x^{k+1} = 0$$

$$\sum_{k=0}^{\infty} k(k-1)A_k x^{k-2} - \sum_{k=3}^{\infty} A_{k-3} x^{k-2} = 0$$

$$2(2-1)A_2 + \sum_{k=3}^{\infty} x^{k-2}\left(k(k-1)A_k - A_{k-3}\right) = 0$$

$\Downarrow$

$2A_2 = 0 \qquad A_{k-3} = k(k-1)A_k \Rightarrow A_k = \frac{A_{k-3}}{k(k-1)} \qquad k=3,\ldots,\infty$

$A_2 = 0$

$A_3 = \frac{A_0}{3\cdot 2} \qquad A_4 = \frac{A_1}{4\cdot 3} \qquad A_5 = 0 \quad A_6 = \frac{A_0}{3\cdot 2\cdot 5\cdot 6} \qquad A_7 = \frac{A_1}{4\cdot 3\cdot 7\cdot 6}$

$A_8 = 0$

$$\boxed{y = A_0\left(1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right) + A_1\left(x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \frac{x^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots\right)}$$

1.2.5 b) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

$$\sum_{K=0}^{\infty}(K)(K-1)A_K x^{K-2} + \sum_{K=0}^{\infty}(K)A_K x^K - \sum_{K=0}^{\infty}A_K x^K = 0$$

$$\sum_{K=0}^{\infty}(K)(K-1)A_K x^{K-2} + \sum_{K=0}^{\infty}(K-1)A_K x^K$$

$$\sum_{K=0}^{\infty}(K)(K-1)A_K x^{K-2} + \sum_{K=2}^{\infty}(K-3)A_{K-2} x^{K-2}$$

$$\sum_{K=2}^{\infty}\left((K)(K-1)A_K + (K-3)A_{K-2}\right)x^{K-2} = 0$$

$$A_K = \frac{(3-K)A_{K-2}}{(K)(K-1)} \quad K=2,3,\dots\to\infty$$

$A_0=A_0, A_1=A_1, A_2=\frac{1\,A_0}{2!}, A_3=0, A_4=\frac{-1\,A_2}{4\cdot 3}=\frac{-A_0}{4!}, A_5=\frac{-2A_3}{}=0$

$$A_6=\frac{-3\,A_4}{6\cdot 5}=\frac{+3A_0}{6!} \quad A_7=0$$

$$A_8=\frac{-5A_6}{8\cdot 7}=\frac{-5\cdot 3\cdot A_0}{8!}$$

$$\boxed{y=A_0\left(1+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{3x^6}{6!}-\frac{5\cdot 3\,x^8}{8!}+\dots\right)+A_1x}$$

2.5c) $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 0$

$$x\sum_0^\infty (k)(k-1)A_k x^{k-2} - \sum_0^\infty (k)A_k x^{k-1} - 4x^3\sum_0^\infty A_k x^k = 0$$

$$\sum_0^\infty (k)(k-1)A_k x^{k-1} - \sum_0^\infty (k)A_k x^{k-1} - \sum_0^\infty 4A_k x^{k+3} = 0$$

$$\sum_{k=0}^\infty (k)(k-2)A_k x^{k-1} - \sum_{k=4}^\infty 4A_{k-4} x^{k-1} = 0$$

$$-1A_1 + 3A_3x^2 + \sum_{k=4}^\infty \left((k)(k-2)A_k - 4A_{k-4}\right)x^{k-1} = 0 \qquad \Rightarrow A_k = \frac{4A_{k-4}}{(k)(k-2)}$$

$A_0 = A_0$, $A_2 = A_2$, $(1)(-1)A_1 \Rightarrow A_1 = 0$. $(3)(1)A_3x^2 \Rightarrow A_3 = 0$

$A_0 = A_0$, $A_1 = 0$, $A_2 = A_2$, $A_3 = 0$, $A_4 = \frac{4A_0}{4\cdot 2} = \frac{A_0}{2}$, $A_5 = 0$, $A_6 = \frac{A_2}{6\cdot 1} = \frac{A_2}{3!}$

$A_8 = \frac{4A_4}{8\cdot 6} = \frac{A_0}{4\cdot 3\cdot 2} = \frac{A_0}{4!}$ | $A_{10} = \frac{4A_2}{10\cdot 8\cdot 3!} = \frac{A_2}{5\cdot 4\cdot 3!} = \frac{A_2}{5!}$

$$y = A_0\left(1 + \frac{x^4}{2!} + \frac{x^8}{4!} + \frac{x^{12}}{6!} + \cdots\right) + A_2\left(x^2 + \frac{x^6}{3!} + \frac{x^{10}}{5!} + \frac{x^{14}}{7!} + \cdots\right)$$

$$= \cosh(x^2) + \sinh(x^2)$$

4.2.6 c) $\left(1-\frac{1}{2}x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

$$\sum_0^\infty (k)(k-1)A_k x^{k-2} - \sum_0^\infty \frac{1}{2}(k)(k-1)A_k x^k + \sum_0^\infty (k)A_k x^k - \sum_0^\infty A_k x^k = 0$$

$$\sum_0^\infty (k)(k-1)A_k x^{k-2} - \sum_2^\infty \frac{1}{2}(k-2)(k-3)A_{k-2}x^{k-2} + \sum_2^\infty (k-2)A_{k-2}x^{k-2} - \sum_2^\infty A_{k-2}x^{k-2} = 0$$

$$\sum_0^\infty (k)(k-1)A_k x^{k-2} - \sum_2^\infty \left((k-2) - \frac{1}{2}(k-2)(k-3) - 1\right)A_{k-2}x^{k-2} = 0$$

$$\sum_{k=2}^\infty \left[(k)(k-1)A_k - \left(1+\frac{1}{2}(k-2)(k-3)-(k-2)\right)A_{k-2}\right]x^{k-2} = 0$$

$$(k)(k-1)A_k = \left(1+\frac{1}{2}(k-2)(k-3)-(k-2)\right)A_{k-2}$$
$$= \left[1 + (k-2)\left(\frac{1}{2}(k-3)-1\right)\right]A_{k-2}$$
$$A_k = \frac{\left(\frac{1}{2}(k^2-5k+6)-k+3\right)A_{k-2}}{(k)(k-1)}$$

$A_0 = A_0,\ A_1 = A_1$

$$A_2 = \frac{(1+0(\sim)A_0}{2} = \frac{A_0}{2}$$

$$A_3 = \left[1+1\left(\frac{1}{2}(0)-1\right)\right]A_1 = 0A_1 = 0$$
$$A_4 = \left[1+2\left(\frac{1}{2}(1)-1\right)\right]A_2 = 0A_2 = 0$$
$$A_5 = \left[1+3\left(\frac{1}{2}(2)-1\right)\right]A_3 = (\sim)0 = 0$$
$\vdots$

$$\boxed{y = A_0\left(1+\frac{1}{2}x^2\right) + A_1 x}$$

4.2.6 a)

$$(x^2+x)\frac{d^2y}{dx^2} - (x^2-2)\frac{dy}{dx} - (x+2)y = 0$$

$$\sum_0^\infty (k)(k-1)A_k x^k + \sum (k)(k-1)A_k x^{k-1} - \sum kA_k x^{k+1} + \sum 2kA_k x^{k-1} - \sum A_k x^{k+1} - \sum 2A_k x^k = 0$$

$$\sum_0^\infty [(k)(k-1)-2]A_k x^k + \sum_0^\infty [(k)(k-1)+2k]A_k x^{k-1} - \sum (k+1)A_k x^{k+1} = 0$$

$$\sum_1^\infty [(k-1)(k-2)-2]A_{k-1} x^{k-1} + \sum_0^\infty ((k)(k-1)+2k)A_k x^{k-1} - \sum_2 (k-1)A_{k-2} x^{k-1} = 0$$

$$-2A_0 + 2A_1 + \sum_2^\infty \left[[(k-1)(k-2)-2]A_{k-1} + [(k)(k-1)+2k]A_k - [k-1]A_{k-2}\right] x^{k-1} = 0$$

$A_0 = A_1$

$$\underbrace{((k)(k-1)+2k)}_{k(k+1)}A_k = (k-1)A_{k-2} - [(k-1)(k-2)-2]A_{k-1}$$

$$A_k = \frac{(k-1)A_{k-2} - [(k-1)(k-2)-2]A_{k-1}}{(k)(k+1)} \quad k=2,3,\dots$$

$$A_2 = \frac{1A_0 - [-2]A_1}{2\cdot 3} = \frac{A_0+2A_0}{3!} = \frac{3A_0}{3!} = \frac{A_0}{2!}$$

$$A_3 = \frac{2A_1 - [(2)(1)-2]A_2}{3\cdot 4} = \frac{2A_1}{3\cdot 4} = \frac{A_0}{3!}$$

$$A_4 = \frac{3A_2 - [(3)(2)-2]A_3}{4\cdot 5} = \frac{3\frac{A_0}{2!} - 4\frac{A_0}{3!}}{4\cdot 5} = \frac{A_0\left(\frac{3}{2!}-\frac{4}{3!}\right)}{4\cdot 5} = \frac{A_0(3\cdot 3! - 4\cdot 2}{2\cdot 3!}$$

$$= \frac{3\cdot 6 - 8}{2\cdot 5!}A_0$$

$$= \frac{5}{5!}A_0$$

$$= \frac{A_0}{4!}$$

$$\boxed{y = A_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)}$$

4.2.6 f)

$$x^2 \frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

$$\sum_{k=0}^{\infty} (k)(k-1) A_k x^k - \sum_{0}^{\infty} (k) A_k x^{k-1} = 0$$

$$\sum_{k=1}^{\infty} (k-1)(k-2) A_{k-1} x^{k-1} - \sum_{0}^{\infty} (k) A_k x^{k-1} = 0$$

$$\sum_{k=1}^{\infty} [(k-1)(k-2) A_{k-1} - k A_k] x^{k-1} = 0$$

$A_0 = A_0$

$$A_k = \frac{(k-1)(k-2) A_{k-1}}{k} \qquad k = 1, 2, \ldots \to \infty$$

$A_1 = 0$

$A_2 = 0$

$A_3 = 0$

$\vdots$

$$\boxed{y = A_0}$$ ■