HW Solutions 7
6, 8, 9, 10, 12

422A/522A
Answers to Homework Problems

Problem 6

If

$$L_{n(t)} = e^t \frac{d^n}{dt^n}(t^n e^{-t})$$

Show that

$$\mathcal{L}\left[L_n(t)\right] = \frac{n!}{s}\left(\frac{s-1}{s}\right)^n$$

Quick Solution

Let

$$f(t) = \frac{d^n}{dt^n}(t^n e^{-t}) \text{ and } \mathcal{L}\left[f(t)\right] = F(s)$$

Therefore

$$\mathcal{L}\left[e^t f(t)\right] = F(s-1)$$

Now let $g(t) = t^n e^{-t}$ and $\mathcal{L}\left[g(t)\right] = G(s)$

and $\mathcal{L}\left[\overbrace{\frac{d^n}{dt^n}(g(t))}^{f(t)}\right] = s^n G(s) \equiv F(s)$

Let $h(t) = e^{-t}$ and $\mathcal{L}\left[h(t) = H(s)\right]$

$\mathcal{L}\left[\overbrace{t^n h(t)}^{g(t)}\right] = (-1)^n \frac{d^n}{ds^n} H(s) = G(s)$

and finally $\mathcal{L}\left[h(t)\right] = H(s) = \mathcal{L}\left[e^{-t}\right] = \frac{1}{s+1}$

1 is the translation property

$$\overset{①\text{-}②}{F(s-1)} = (s-1)^n\, G(s-1) = (s-1)^n(-1)^n \frac{d^n}{ds^n} H(s-1) = (s-1)^n(-1)^n \frac{d^n}{ds^n} = \frac{1}{s}$$

$$= \frac{n!}{s}\left(\frac{s-1}{s}\right)^n$$

Problem 8

Show

$$\mathcal{L}\left[\left(t\frac{d}{dt}\right)^n f(t)\right] = (-1)^n \frac{d}{ds}\left(s\frac{d}{ds}\right)^{n-1} (sF(s)) \quad n > 0$$

Note: Common mistake is to assume that $(t\frac{d}{dt})^n f = t^n \frac{d^n}{dt^n} f$!

$n = 1 \quad \mathcal{L}\underbrace{\left[\left(t\frac{d}{dt}\right)^n f(t)\right]}_{g(t)} = \mathcal{L}[t\, g(t)] = \frac{d}{ds}\, G(s)$

But

$$\mathcal{L}[g(t)] = \mathcal{L}\left[\frac{d}{dt}\, f(t)\right] = S\, F(s) \equiv G(s)$$

Therefore

$$\mathcal{L}\left[\left(t\frac{d}{dt}\right) f(t)\right] = -\frac{d}{ds} S\, F(s) = H(s)$$

$n = 2 \quad \mathcal{L}\left[\left(t\frac{d}{dt}\right)^2 f(t)\right] = \mathcal{L}\left[\left(t\frac{d}{dt}\right) h(t)\right]$

Where

$$h(t) = t\frac{d}{dt} f(t) \text{ and } \mathcal{L}[h(t)] = \frac{d}{ds}(sF(s))$$

Therefore,

$$\mathcal{L}\left[\left(t\frac{d}{dt}\right) h(t)\right] = \mathcal{L}[tu(t)] = -\frac{d}{ds} u(s)$$

Where

$$U(t) = \frac{d}{dt} h(t)$$

and $\mathcal{L}[u(t)] = \mathcal{L}\left[\frac{d}{dt} h(t)\right] = sH(s) = u(s)$

Therefore,

$$\begin{aligned}\mathcal{L}\left[t\frac{d}{dt} h(t)\right] &= -\frac{d}{ds}(sH(s)) = -\frac{d}{ds}\left[s\left(-\frac{d}{ds}(sF(s))\right)\right] \\ &= +\frac{d}{ds}\left(s\frac{d}{ds}\right)(sF(s))\end{aligned}$$

By induction result follows for general n.

7. (a)

$$\begin{aligned}\mathcal{L}[f(at)] &= \int_0^\infty e^{-st} f(at)\, dt \\ &= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}\tau} f(\tau)\, d\tau \qquad \tau = \frac{t}{a},\ dt = \frac{d\tau}{a} \\ &= \frac{1}{a} F\left(\frac{s}{a}\right)\end{aligned}$$

(b)

$$\mathcal{L}^{-1}\{F(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right)$$

$$\frac{1}{a}\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\}. = \frac{1}{a}\int_0^\infty e^{-st}f\left(\frac{t}{a}\right)dt \qquad \tau = \frac{t}{a},\ dt = a d\tau$$

$$= \int_0^\infty e^{-sa\tau}f(\tau)\,d\tau = \bar{F}(as)$$

Problem 9

$$f(t) = F(t) \qquad 0 > t > a \text{ and } f(t) = f(t+a)$$

i.e. periodic function

$$\mathcal{L}[f(t)] = \int_0^a e^{-st}F(t)\,dt + \int_a^{2a} e^{-st}F(t)\,dt \int_{2a}^{3a} e^{-st}F(t)\,dt + \cdots$$

$$= \int_0^a e^{-st}F(t)\,dt + \int_0^a e^{-s(t+a)}F(t)\,dt + \int_0^a e^{-s(t+2a)}F(t)\,dt + \cdots$$

$$= \int_0^a e^{-st}F(t)\,dt\ \left[1 + e^{as} + e^{-2as} + \cdots\right]$$

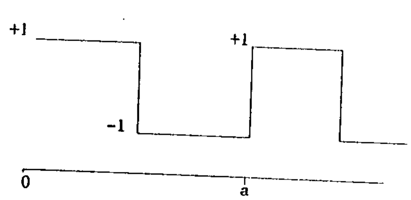
$$= \frac{\int_0^a e^{-st}F(t)\,dt}{1-e^{-as}} \qquad s > 0$$

Problem 10

Square-wave function

$$F(t) = 1 \quad 0 < t < \frac{a}{2}$$
$$F(t) = -1 \quad \frac{9}{2} < t < a$$

and periodic with period a $\quad \frac{a}{2} < t < a$

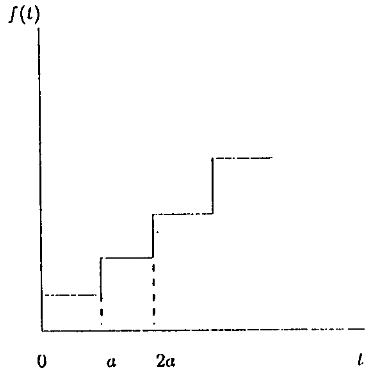


$$\frac{\int_0^{a/2} e^{-st}\,dt - \int_{\frac{a}{2}}^a e^{-st}}{1-e^{-as}}\,dt = \frac{1}{s}\frac{\left(1-e^{-as/2}\right)^2}{1-e^{-as}} = \frac{1}{s}\frac{\left(1-e^{-as/2}\right)}{\left(1+e^{-as/2}\right)}$$

$$= \frac{1}{s}\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}} = \frac{1}{s}\tanh\ (as/4)$$

Problem 12

Staircase function



$$\int_0^\infty e^{-st} f(t)t\,dt = b\int_0^a e^{-st}dt + 2b\int_a^{2a} e^{-st}dt + \int_{2a}^{3a} e^{-st}dt + \cdots$$

$$= b\left[-\frac{1}{s}e^{-st}\Big|_0^a - \frac{2}{s}e^{-st}\Big|_0^{2a} - \frac{3}{s}e^{-st}\Big|_{2a}^{3a} + \ldots\right]$$

$$= \frac{b}{s}\left[-e^{as} + 1 - 2e^{-2as} + 2e^{-as} - 3e^{-3as} + 3e^{-2as} + \cdots\right]$$

$$= \frac{b}{s}\left[1 + e^{-2as} + e^{-as} + \cdots\right]$$

Problem 13

(a) If $F(s) = \mathcal{L}[f(t)]$, and if an interchange of order of integration is valid, then

$$\int_s^\infty F(v)\,dv = \mathcal{L}\frac{f(t)}{t}$$

$$F(v) = \int_0^\infty e^{-vt} f(t)\,dt$$

$$\int_s^\infty \int_0^\infty e^{-vt} f(t)\,dt\,dv = \int_0^\infty \left[\int_s^\infty e^{-vt}dv\right] f(t)\,dt$$

$$= \int_0^\infty \left[-\frac{e^{-vt}}{t}\right]_s^\infty f(t)\,dt = \int_0^\infty e^{-st}\frac{f(t)}{t}dt = \mathcal{L}\left[f\frac{(t)}{t}\right]$$

(b)

$\mathcal{L}\left[\frac{\sin t}{t}\right] = \cot^{-1} s$ i.e. $f(t) = \sin t \quad \mathcal{L}[f(t)] = F(v) = \frac{1}{v^2+1}$

Therefore,

$$\mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty F(v)\,dv = \int_s^\infty \frac{dv}{v^2+1} = -\cot^{-1} v\Big|_s^\infty = \cot^{-1} s$$

$$\mathcal{L}\left[\frac{1-e^{-t}}{t}\right] = \log\frac{[s+1]}{s}, \quad f(t) = 1 - e^t\,\mathcal{L}[f(t)] = \frac{1}{v} - \frac{1}{v+1}$$

Therefore,

$$\int_s^\infty dv\left[\frac{1}{v} - \frac{1}{v+1}\right] = \log\frac{v}{v+1}\Big|_s^\infty = -\log\frac{s}{s+1} = \log\frac{s+1}{s}$$