HW 10 Solutions   /40

1) a) $w = \sin(z)$ $w' = \cos z$ has zeroes at $z = (2n+1)\frac{\pi}{2}$, n integer ~~$z = \pi/2$~~, ~~...~~

b) $w = \frac{1}{z}$ $w' = -\frac{1}{z^2}$ fails to be analytic at $z=0$

c) $w = z^2 - z$ $w' = 2z - 1$ $=0$ when $z = 1/2$

d) $w = 1 - \cos(z)$ $w' = \sin z$ $=0$ when $z = \pm n\pi$, $n = 0, 1, 2, ..$

2) a) $w = z^3 \sin(z)$

$w' = 3z^2 \sin(z) + z^3 \cos(z)$ $=0$ at $z=0$

$w'' = 6z \sin z + 3z^2 \cos z + 3z^2 \cos z - z^3 \sin z$ $=0$ at $z=0$

$w''' = 6 \sin z + 6z \cos z$

$w'''' = 6 \cos z + 6 \cos z$ ... $\neq 0$ at $z=0$

$\therefore$ Zero of order 3 $\Rightarrow$ angle increases by $(k+1) = 4$!

b) $w = z - \sin z$

$w' = 1 - \cos z$ $=0$ at $z=0$

$w'' = \sin z$ $=0$ at $z=0$

$w''' = \cos z$ $\neq 0$ at $z=0$ $\Rightarrow k=2$

Increases angle by factor of 3.

c) $w = e^z - z$

$w' = e^z - 1$ $=0$ at $z=0$

$w'' = e^z$ $\neq 0$ at $z=0$ doubles angle.

d) $w = e^{z^2} - \cos z$

$w' = 2z e^{z^2} + \sin(z) \quad = 0 \quad \text{at } z = 0$

$w'' = 2e^{z^2} + 4z^2 e^{z^2} + \cos z \neq 0 \text{ at } z = 0$

doubles angle.

3) $W = \dfrac{Z-1}{Z+1}$ maps the right-half-plane onto the unit disk.

By substituting the inverse mapping

$$Z = \frac{1+W}{1-W}$$

we obtain the equivalent problem of finding the number of roots of the equation

$$g(w) = w^4 + 3w^3 + 8w^2 - 2w + 1 = 0$$

lying in $|w| < 1$.

Let $f(w) = 8w^2$ $\qquad$ $f(w)$ has 2 zeros

$|g(w) - f(w)| \leq 7 < 8|w|^2 = 8$

$\therefore$ original fcn $p(z)$ has 2 zeroes in the right-half-plane.

4) When $z = 1$, $w = 0$

$z = -1$, $w = \infty$

$z = 0$, $w = -i$

$\therefore$ unit disk is mapped to lower half plane.