Math 424
Solution Homework 9

1. a) Let $g(z) = z^5 + 8z + 10$ and $f(z) = 8z + 10$. single root at $z = -5/4$ — $|z| > 1$ so no zero inside unit circle.

$|f(z)| < 8|z| + 10 = 18$ on $|z| = 1$
$|g(z) - f(z)| < |z|^5 = 1$ on unit circle. No roots inside unit circle.

b) Let $g(z) = z^8 - 2z^5 + z^3 - 8z^2 + 3$ and $f(z) = -8z^2$ and note that

$$| g(z) - f(z) | < 8 = | f(z) | \text{ on } | z | = 1$$

Hence, there are two roots of g inside $| z | = 1$.

c) Let $g(z) = z^6 + 3z^5 - 2z^2 + 2z - 9$ and $f(z) = 2z - 9$. $f(z)$ has no root inside unit circle and

$|f(z)| \le 2|z| + 9 = 11$ on $|z| = 1$ and $|g(z) - f(z)| \le |z|^6 + 3|z|^5 + 2|z|^2 = 6$ on $|z| = 1$.

No roots inside unit circle.

d) Let $f(z) = -7z^6$. There are six roots inside the unit circle.

2. a) Let $f(z) = z^5$. $|f(z)| \le |z|^5 = 32$ on $|z| = 2$. $f(z)$ has 5 roots inside $|z| = 2$.
$|g(z) - f(z)| \le 8|z| + 10 = 26$ on $|z| = 2$.
5 roots inside $|z| = 2$.

b) Let $f(z) = z^8$ $|f(z)| \le |z|^8 = 256$ on $|z| = 2$. 8 roots inside $|z| = 2$
$|g(z) - f(z)| \le 2|z|^5 + |z|^3 + 8|z|^2 + 3 = 107$ on $|z| = 2$
8 roots inside $|z| = 2$

c) Let $f(z) = z^6 + 3z^5 = z^5(z + 3)$. $|f(z)| \le |z|^5(|z| + 3) = 160$ on $|z| = 2$. 5 roots inside and one outside $|z| = 2$

$|g(z) - f(z)| \le 2|z|^2 + 2|z| + 9 = 21$ on $|z| = 2$.

5 roots inside $|z| = 2$.

3. Two roots. Let
$f(z) = 3z^4 + 7z^2 + 2 = (3z^2 + 1)(z^2 + 2)$

and $g(z)$ be given polynomial. Then,

$|g(z)-f(z)| \le 2|z||3z^2+1| < |f(z)|$,

Since $2|z| < |z^2+2|$ for all z.

4. Choose $g(z)=z^7-5z^3+12$ *and* $f(z)=12$. Inside the unit circle

$|z|=1, \quad |g(z)-f(z)|=|z^7-5z^3| \le |z|^7+5|z|^3=11<12$

But f(z) has no zeroes, so no roots inside the unit circle.

Set |z|=2 and choose

$f(z)=z^7 \quad \rightarrow |g(z)-f(z)|=|-5z^3+12| \le 5|z|^3+12=19<|f(z)|=2^7$

But f(z) has 7 roots inside |z|=2. Therefore, all roots contained within the annulus 1<|z|<2.