MATH 424
HOMEWORK 8

1. a) $$\int_{-\infty}^{\infty} \frac{\sin \pi x \, dx}{x(x^4-1)} = \text{Im } PV \int_{-\infty}^{\infty} \frac{e^{i\pi x} \, dx}{x(x-1)(x+1)(x^2+1)}$$

$$= \text{Im } 2\pi i \left[\text{Res}_i \frac{e^{i\pi z}}{z(z^4-1)} + \frac{1}{2} (\text{Res}_0 + \text{Res}_1 + \text{Res}_{-1}) \frac{e^{i\pi z}}{z(z^4-1)} \right]$$

$$= \text{Im } 2\pi i \left[\frac{e^{-\pi}}{4} + \frac{1}{2}\left(-1 + \frac{1}{2}\cos\pi\right) \right] = \frac{\pi}{2}(e^{-\pi} - 3)$$

b) $$\int_{-\infty}^{\infty} \frac{\sin x}{x} \frac{x^2+a^2}{x^2+b^2} \, dx = \text{Im } PV \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \frac{x^2+a^2}{x^2+b^2} \, dx$$

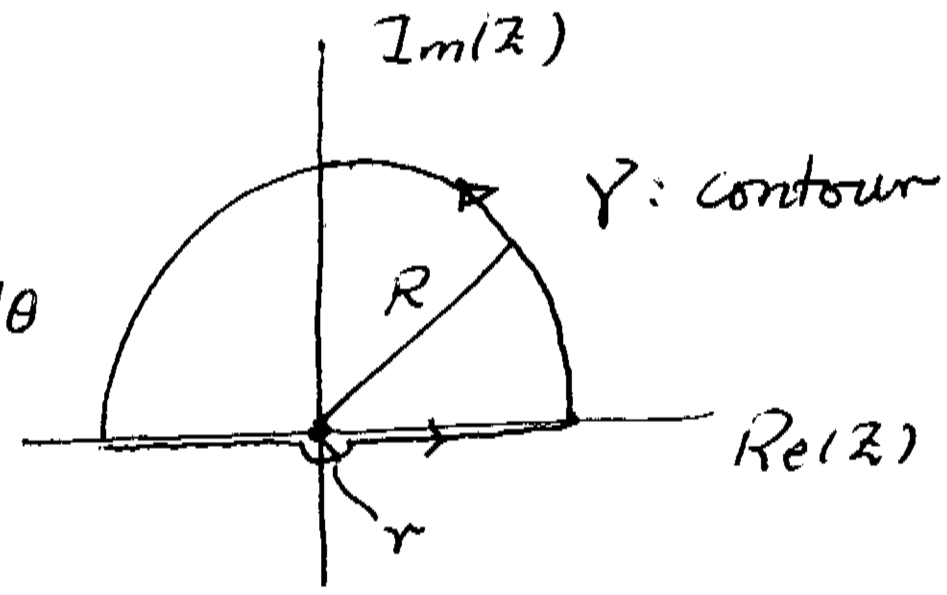
$$= \text{Im } 2\pi i \left[\text{Res}_{bi} \frac{z^2+a^2}{z^2+b^2} \frac{e^{iz}}{z} + \frac{1}{2} \text{Res}_0 \frac{z^2+a^2}{z^2+b^2} \frac{e^{iz}}{z} \right]$$

$$= \text{Im } 2\pi i \left[\frac{(b^2-a^2)e^{-b}}{2b^2} + \frac{a^2}{2b^2} \right] = \frac{\pi}{b^2}\left[a^2 + e^{-b}(b^2-a^2)\right]$$

c) $$\frac{1}{2} \text{Im } 2\pi i \left[\text{Res}_{bi} \frac{e^{iaz}}{z(z^2+b^2)^2} + \frac{1}{2} \text{Res}_0 \frac{e^{iaz}}{z(z^2+b^2)^2} \right]$$

$$= \text{Im } \pi i \left[\lim_{z \to bi} e^{iaz} \left(\frac{iaz^2 - abz - 3z - bi}{z^2(z+bi)^3} \right) + \frac{1}{2b^4} \right]$$

2. For $t > 0$, use the contour on the right.



Then, for $z = Re^{i\theta}$

$$\left| \frac{1}{2\pi} \int_0^{\pi} e^{itRe^{i\theta}} d\theta \right| < \frac{1}{\pi} \int_0^{\pi/2} e^{-tR\sin\theta} d\theta$$

$$\leq \frac{1}{\pi} \int_0^{\pi/2} e^{-tR\theta/\pi} d\theta \to 0 \text{ as } R \to +\infty$$

For r sufficiently small,

$$\left| \frac{1}{2\pi} \int_{-\pi}^{0} e^{itre^{i\theta}} d\theta - \frac{1}{2} \right| = \left| \frac{1}{2\pi} \int_{-\pi}^{0} (e^{itre^{i\theta}} - 1) \, d\theta \right| < \epsilon\pi$$

Hence,

$$PV \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{itx}}{x} dx = \frac{1}{2\pi} \oint_{\gamma} \frac{e^{itz}}{z} dz - \frac{1}{2\pi} \int_{-\pi}^{0} e^{itre^{i\theta}} d\theta$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

For $t<0$, use the contour that consists of all points conjugate those on γ; $t=0$, use defination of PV of real integral in class.

3. a) Let $x = e^y$. Note that

$$\int_{-\infty}^{\infty} \frac{e^{ay}}{1+be^{-y}}\,dy = \int_0^{\infty} \frac{x^a\,dx}{x+b}$$

b) For $a=0$, or $\theta=0$, interpret the answer as a limit

Then, for $\theta \neq 0$, and $a \neq 0$

$$\int_0^{\infty} \frac{x^a\,dx}{x^2+2x\cos\theta+1} = \frac{2\pi i}{1-e^{2\pi i a}} \sum_{z\neq 0} \operatorname{Res} \frac{z^a}{(z+e^{i\theta})(z+e^{-i\theta})}$$

$$= \frac{2\pi i}{1-e^{2\pi i a}} \left[\frac{\sin\theta a}{\sin\theta} \right] e^{-\pi i a} = \frac{\pi}{\sin\pi a} \left[\frac{\sin\theta a}{\sin\theta} \right].$$

c) For $a=0, 1$, interpret the answer as a limit. For $a \neq 0, 1$

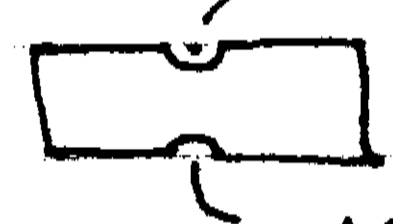
$$\int_0^{\infty} \frac{x^a}{x^3+b^3}\,dx = \frac{2\pi i}{1-e^{2\pi i a}} \left[\operatorname{Res}_{-b} + \operatorname{Res}_{b e^{i\pi/3}} + \operatorname{Res}_{b e^{-i\pi/3}} \right] \frac{z^a}{z^3+b^3}$$

4. Use $f(z) = z e^{az} / \sinh z$

(Hint: use contour with vertical sides of length πi, $f(z) = z/\sinh z$. the identity $\sinh(x+\pi i) = -\sinh x$, and in-class example)

HW 9 Math 421

4 $$\int_0^\infty \frac{x \cos ax}{\sinh x} = \frac{\pi^2}{4} \operatorname{Sech}^2 \frac{a\pi}{2}, \quad a \text{ real}$$

Soln: Use rectangular contour in class 

$$\oint_C \frac{z e^{iaz}}{\sinh z}\,dz = \pi i\left[\operatorname*{Res}_{z=0} \frac{z e^{iaz}}{\sinh z} + \operatorname*{Res}_{z=\pi i} \frac{z e^{iaz}}{\sinh z}\right]$$

$$= \pi i\left[0 - \pi i e^{-a\pi}\right] = \pi^2 e^{-a\pi}$$

But $$\oint_C \frac{z e^{iaz}}{\sinh z}\,dz \to \int_{-\infty}^{\infty} \frac{x e^{iax}}{\sinh x}\,dx + e^{-a\pi}\int_{-\infty}^{\infty} \frac{(x+\pi i) e^{iax}}{\sinh x}\,dx$$

as contributions on sides $\to 0$ as $R \to \infty$

$$\therefore (1+e^{-a\pi})\int_{-\infty}^{\infty} \frac{x e^{iax}}{\sinh x}\,dx + \pi i e^{-a\pi}\int \frac{e^{iax}}{\sinh x}\,dx = \pi^2 e^{-a\pi}$$

must evaluate this integral on same contour

$$\oint_C \frac{e^{iaz}}{\sinh z}\,dz = \pi i\left[\operatorname*{Res}_{z=0} \frac{e^{iaz}}{\sinh z} + \operatorname*{Res}_{z=\pi i} \frac{e^{iaz}}{\sinh z}\right]$$

$$(1+e^{-a\pi})\int_{-\infty}^{\infty} \frac{e^{iax}}{\sinh x}\,dx = \pi i e^{-a\pi}(1-e^{-a\pi}) \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{iax}}{\sinh x}\,dx = \frac{\pi i e^{-a\pi}(1-e^{-a\pi})}{1+e^{-a\pi}}$$

$$\int_{-\infty}^{+\infty} \frac{x e^{iax}}{\sinh x}\,dx = \frac{\pi^2 e^{-a\pi}(1-e^{-a\pi})}{(1+e^{-a\pi})^2} + \frac{\pi^2 e^{-a\pi}}{(1+e^{-a\pi})} = \pi^2 \frac{e^{-a\pi}}{(1+e^{-a\pi})^2}$$

$$= \frac{\pi^2}{2} \operatorname{Sech}^2 \frac{a\pi}{2}$$