Math 424
Homework Solutions 7

1. Find the residue at all of the singularities in the complex plane for the following functions:

Solution:
(a) $f(z) = z\exp(\frac{1}{z}) = z\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right)$. Therefore, $c_{-1} = \frac{1}{2}$

b)
$f(z) = (z-1)\exp(\frac{1}{z}) = z\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right) - \left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right) = z + (\frac{1}{2}$
$\therefore c_{-1} = -\frac{1}{2}$

2. Evaluate the integrals

Solution:
a) Simple pole at $z = \pm i$, pole of order n at $z = 0$. No poles outside the circle centered at $z = 0$ and of radius 2. Use outside theorem i.e

$$\int_{|z-1|=2} \frac{dz}{z^n(z^2+1)} = -2\pi i \sum_{outside} \text{Res}\left[\frac{1}{z^n(z^2+1)}\right] = 0$$

b) $\int_{|z-\frac{1}{2}|=1} \frac{\sin(z)}{(z^3+z)}dz = \int_{|z-\frac{1}{2}|=1} \frac{\sin(z)}{z(z^2+1)}dz$. Simple poles at $z = 0, \pm i$. Only pole at $z = 0$ lies inside unit circle centered at
$z = \frac{1}{2}$. $\therefore \text{Res}_{z=0}\left[\frac{\sin(z)}{z(z^2+1)}\right] = 0$. So $\int_{|z-\frac{1}{2}|=1} \frac{\sin(z)}{(z^3+z)}dz = 0$.

3. Evaluate the definite integrals
Solution:

a) $$\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{1}{i}\int_{|z|=1} \frac{dz}{z\left(\frac{a^2}{4}(z+1/z)^2 - \frac{a^2}{4}(z-1/z)^2\right)} = -4i\int_{|z|=1} \frac{zdz}{((a^2-b^2)z^4 + 2(a^2+b^2)}$$

$= \frac{-4i}{a^2-b^2}\int_{|z|=1} \frac{zdz}{\left(z^2+\frac{a+b}{a-b}\right)\left(z^2+\frac{a-b}{a+b}\right)}$. Since $\frac{a-b}{a+b} < 1$ and $\frac{a+b}{a-b} > 1$, the 2 simple poles at $z = \pm i\sqrt{\frac{a-b}{a+b}}$ lie inside the unit circle.

Therefore $\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} =$

$$2\pi i \frac{-4i}{a^2-b^2}\left[\lim_{z\to i\sqrt{\frac{a-b}{a+b}}} \frac{z}{\left(z^2 + \frac{a+b}{a-b}\right)\left(z + i\frac{a-b}{a+b}\right)} + \lim_{z\to i\sqrt{\frac{a-b}{a+b}}} \frac{z}{\left(z^2 + \frac{a+b}{a-b}\right)\left(z - i\frac{a-b}{a+b}\right)}\right]$$

$$= \frac{8\pi}{a^2-b^2}\left[\frac{a^2-b^2}{4ab}\right] = \frac{2\pi}{ab}$$

b) $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2}$

$= -i\int_{|z|=1} \frac{dz}{(-az^2 + (a^2+1)z - a)} = -i\int_{|z|=1} \frac{dz}{(az-1)(z-a)} = \frac{i}{a}\int_{|z|=1} \frac{dz}{(z-\frac{1}{a})(z-a)}$. Two simple poles at

$z = a$ and $z = \frac{1}{a}$.

For $|a| < 1$ the pole at $z = a$ lies inside the unit circle and

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} = 2\pi i\left(\frac{i}{a}\right)\lim_{z\to a}\frac{1}{z-\frac{1}{a}} = \frac{2\pi}{1-a^2}$$

For $|a| > 1$ the pole at $z = \frac{1}{a}$ lies inside the unit circle and

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} = 2\pi i\left(\frac{i}{a}\right)\lim_{z\to\frac{1}{a}}\frac{1}{z-a} = \frac{2\pi}{a^2-1}$$

4. Evaluate the following indefinite integrals

a) $\int_0^{\infty} \frac{x^2dx}{(x^2+a^2)^2}$

Use the formula for improper integrals $\int_{-\infty}^{\infty} f(x)e^{ibx}dx = 2\pi i\sum_{y>0}\text{Res}[f(z)e^{ibz}]$ where $b = 0$ and the contour is a semicircle enclosing all poles in

the UHP. $\therefore$ $f(z) = \frac{z^2}{(z^2+a^2)^2} = \frac{z^2}{(z+ia)^2(z-ia)^2}$ and there are poles of order 2 at

$z = \pm ia,\ a > 0$. Only the second order pole at

$z = ia$ is contained within the contour.

$\therefore$ $\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i\text{Res}_{z=ia}\frac{d}{dz}\left[\frac{z^2}{(z+ia)^2}\right] = \frac{\pi}{2a}$. Since the integrand is even and taking the

real part we get $\int_0^{\infty} \frac{x^2dx}{(x^2+a^2)^2} = \frac{\pi}{4a}$

b) $\int_{\infty}^{\infty} \frac{x^3\sin(ax)dx}{(x^2+b^2)^2},\ a,b > 0$

Use the formula for improper integrals $\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i \sum_{y>0} \text{Re}\,s[f(z)e^{iaz}]$ where $a > 0$ and

the contour is a semicircle enclosing all poles in
the UHP. $\therefore\ f(z) = \dfrac{z^3}{(z^2+b^2)^2} = \dfrac{z^2}{(z+ib)^2(z-ib)^2}$ and there are poles of order 2 at
$z = \pm ia,\ a > 0$. Only the second order pole at
$z = ib$ lies inside the contour.

$$\therefore \int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i\,\text{Re}\,s_{z=ib}\frac{d}{dz}\left[\frac{e^{iaz}z^3}{(z+ib)^2}\right] = \frac{\pi}{2}e^{-ab}(2-ab)$$