MATH 424

SOLUTION HOMEWORK 6

1. a) Observe that $(\cos z)^{(4n)} = -(\cos z)^{(4n+2)} = \cos z$ and $(\cos z)^{(4n+1)} = -(\cos z)^{(4n+3)} = -\sin z$. Evaluate at $z = 0$ and apply Taylor's theorem.

   b) $(\cosh z)^{(2n)} = \cosh z$, $(\cosh z)^{(2n+1)} = \sinh z$ and $\cosh 0 = 1$, $\sinh 0 = 0$.

2. a) $\displaystyle\sum_{n=0}^{\infty} \frac{(z+1)^n}{2^{n+1}}$, $|z+1| < 2$, since $f^{(n)}(z) = \dfrac{n!}{(1-z)^{n+1}}$.

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (z-\pi/2)^{2n-1}}{(2n-1)!}$, $|z| < \infty$.

   c) $\displaystyle i\pi - \sum_{n=1}^{\infty} \frac{(z+1)^n}{n}$, $|z+1| < 1$, since

   $f^{(n)}(z) = -(n-1)! z^{-n}$.

3. a) $$z^2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots - 1\right) = -\frac{z^4}{2!} + \frac{z^6}{4!} - \ldots,$$

   so the order of the zero is 4.

   b) $f'(z) = 1 - \sec^2 z \Big|_{z=0} = 0,$

   $f''(z) = -2\sec^2 z \ \tan z \Big|_{z=0} = 0,$

   $f'''(z) = -4\sec^2 z \ \tan^2 z - 2\sec^4 z \Big|_{z=0} = 1,$

   so the order of the zero is 3.

4. a) $$\frac{1}{z(z^2-1)} = \frac{-1}{z} - \frac{1/2}{1-z} + \frac{1/2}{1+z}$$

   $$= \frac{-1}{z} - \frac{1}{2}\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}(-z)^n = -\sum_{n=0}^{\infty} z^{2n-1}.$$

   b) $$\frac{-1}{1-(1-z)} - \frac{1/2}{1-z} + \frac{1/2}{2-(1-z)}$$

   $$= -\sum_{n=0}^{\infty}(1-z)^n - \frac{1/2}{1-z} + \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1-z}{2}\right)^n$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}(1-2^{-n})(z-1)^{n-2}.$$

5\. a) $$\frac{z}{(z+2)(z-1)} = \frac{2/3}{z+2} + \frac{1/3}{z-1}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-z}{2}\right)^{n} - \frac{1}{3}\sum_{n=0}^{\infty} z^{n}$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}[1-(-2)^{-n}]z^{n}.$$

b) $$\frac{2/3}{z+2} + \frac{1/3}{(z+2)-3} = \frac{2/3}{z+2} - \frac{1/9}{1-(z+2)/3}$$

$$= \frac{2/3}{z+2} - \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{z+2}{3}\right)^{n}.$$

c) $$\frac{2}{3z}\left(\frac{1}{1+2/z}\right) + \frac{1}{3z}\left(\frac{1}{1-1/z}\right)$$

$$= \frac{1}{3z}\left[2\sum_{n=0}^{\infty}\left(\frac{-2}{z}\right)^{n} + \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}\right].$$

6\. a) Expand

$$\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1}$$

$$= \frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)!}\left(\sum_{n=0}^{\infty}(1-z)^{k}\right)^{2n+1}.$$

b) Expand

$$\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)!}\left(\frac{1}{z(z-1)}\right)^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)!}\left[\frac{1}{z-1}\sum_{k=0}^{\infty}(1-z)^{k}\right]^{2n+1}.$$