MATH 424

SOLUTION HOMEWORK 5

1. $|Log\ z| \leq \log R + \frac{\pi}{2}$ on γ, so that

$$\left|\int_\gamma \frac{Log\ z}{z^2} dz\right| \leq (\log R + \frac{\pi}{2}) \int_{-\pi/2}^{\pi/2} \frac{dt}{R} = \frac{\pi}{R}(\log R + \frac{\pi}{2}).$$

2. By Cauchy's theorem for the derivative,

$$|f^{(n)}(z)| = \frac{n!}{2\pi} \left| \int_{|z|=R} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right|$$

$$\leq \frac{Mn!}{2\pi} \int_0^{2\pi} \frac{R\ dt}{(R - |z|)^{n+1}} = \frac{MRn!}{(R - |z|)^{n+1}}$$

3. $\frac{d^n}{dz^n}[(z^2 - 1)^n] = \frac{n!}{2\pi i} \int_\gamma \frac{(\zeta^2 - 1)^n}{(\zeta - z)^{n+1}} d\zeta$

4. Let $P(z) = (z - r_1)^{k_1} \cdot \ldots \cdot (z - r_n)^{k_n} Q(z)$, where $r_1, \ldots, r_n$ lie inside γ and the roots of $Q(z)$ lie outside of γ. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{P'(z)dz}{P(z)} = \frac{1}{2\pi i} \int_\gamma \Big[\frac{k_1}{z - r_1} + \ldots$$

$$+ \frac{k_n}{z - r} + \frac{Q'(z)}{Q(z)}\Big] dz$$

$$= k_1 + \ldots + k_n,$$

since Q'/Q is analytic inside and on γ.