MATH 424 / 524

Homework #1

1 a) $(2+3i)(4+i) = 2\times 4 + 2\times i + 3i\times 4 + 3i\times i$

$= 5+14i$

b) $(8+bi)^2 = 64-b^2+16bi$

c) $\frac{3}{1+i} = \frac{3}{1+i}\cdot\frac{1-i}{1-i} = \frac{3-3i}{2}$

$1+\frac{3}{1+i} = 1+\frac{3-3i}{2} = \frac{5-3i}{2}$

$\left(1+\frac{3}{1+i}\right)^2 = \frac{1}{4}(5-3i)^2 = \frac{16-30i}{4} = 4-\frac{15}{2}i$

2 a) $\frac{z+1}{2z-5} = \frac{(x+1)+iy}{(2x-5)+i2y}\cdot\frac{(2x-5)-i2y}{(2x-5)-i2y}$

$= \frac{(x+1)(2x-5)+2y^2}{(2x-5)^2+4y^2} + i\,\frac{-7y}{(2x-5)^2+4y^2}$

Real part $= \frac{(x+1)(2x-5)+2y^2}{(2x-5)^2+4y^2}$

Imag part $= \frac{-7y}{(2x-5)^2+4y^2}$ [Note: There is no 'i' in imaginary part]

b) $z^3 = (x+iy)^3 = x^3+3x^2(iy)+3x(iy)^2+(iy)^3$

$= (x^3-3xy^2)+i\,(3x^2y-y^3)$

Real part $= x^3-3xy^2$ Imag. Part $= 3x^2y-y^3$

2 c) $$\frac{1}{z^2} = \frac{1}{(x+iy)^2} \cdot \frac{(x-iy)^2}{(x-iy)^2}$$

$$= \frac{x^2-y^2-i\,2xy}{(x^2+y^2)^2}$$

Real part $= \dfrac{x^2-y^2}{(x^2+y^2)^2}$ Imag part $= -\dfrac{2xy}{(x^2+y^2)^2}$

3) $$\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|$$

For $n=1$ $\left|\sum_{k=1}^{1} z_k\right| = |z_k|$, hence the inequality holds.

Let the inequality be true for $n=m$

$$\left|\sum_{k=1}^{m} z_k\right| \le \sum_{k=1}^{m} |z_k|$$

$$\left|\sum_{k=1}^{m+1} z_k\right| = \left|\sum_{k=1}^{m} z_k + z_{m+1}\right|$$

$$\le \left|\sum_{k=1}^{m} z_k\right| + |z_{m+1}|$$ using Triangle Inequality

$$\le \sum_{k=1}^{m} |z_k| + |z_{m+1}|$$

$$= \sum_{k=1}^{m+1} |z_k|$$

By induction the inequality holds for all $n \geq 1$

4) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\, z_1 \bar{z}_2$

$(|z_1| + |z_2|)^2 = |z_1|^2 + |z_2|^2 + 2|z_1 \bar{z}_2|$

~~Let~~ $z_1 \bar{z}_2 = |z_1 \bar{z}_2| \cos\theta + i\, |z_1 \bar{z}_2| \sin\theta$, where $\theta = \arg(z_1 \bar{z}_2)$

$|z_1 + z_2| = |z_1| + |z_2|$ if and only if.

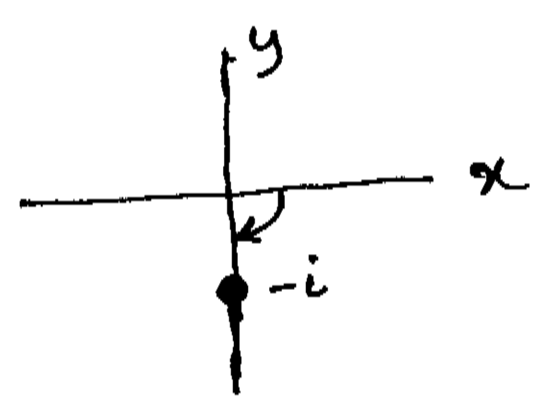
$$\mathrm{Re}(z_1 \bar{z}_2) = |z_1 \bar{z}_2|$$

$$|z_1 \bar{z}_2| \cos\theta = |z_1 \bar{z}_2|$$

$$\therefore \quad \theta = \arg(z_1 \bar{z}_2) = 0$$

$$\arg z_1 - \arg z_2 = 0.$$

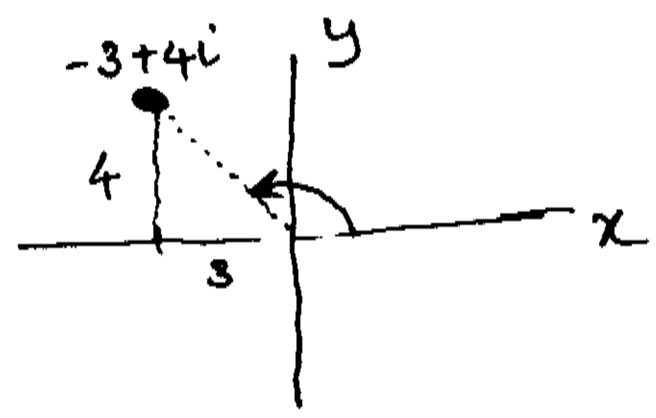
5) a)



$|-i| = 1 \quad \arg(-i) = -\pi/2$

$$-i = e^{-i\pi/2}$$

b)



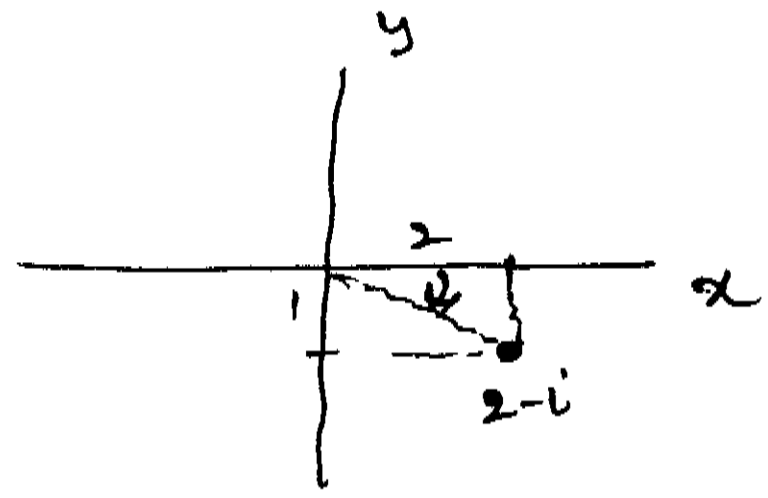
$|-3+4i| = 5 \quad \arg(-3+4i) = \pi - \arctan(4/3)$

$= 2.2143$

$$-3+4i = 5e^{i\,2.2143}$$

NOTE: $\mathrm{Arg}(-3+4i) = \pi - \arctan(4/3)$
$\mathrm{Arg}(3-4i) = -\arctan(4/3)$

5 c)



$$|2-i| = \sqrt{5} \qquad \arg(2-i) = -\arctan(1/2) = -0.4636$$

$$2-i = \sqrt{5}\, e^{-i\,0.4636}$$

6 a)

$$1+i = \sqrt{2}\, e^{i\pi/4}$$

$$(1+i)^{29} = (\sqrt{2})^{29} e^{i\,29\pi/4}$$

$$= (\sqrt{2})^{29} e^{i(6\pi + 5\pi/4)}$$

$$= 2^{29/2} \frac{-1-i}{\sqrt{2}}$$

$$= -2^{14}(1+i)$$

b)

$$-1-i = \sqrt{2}\, e^{-i\,3\pi/4}$$

$$(-1-i)^{36} = (\sqrt{2})^{36} e^{-i\,27\pi}$$

$$= -2^{18}$$

c)

$$\sqrt{3}+i = 2\, e^{i\pi/6}$$

$$(\sqrt{3}+i)^{15} = 2^{15} e^{i\,15\pi/6}$$

$$= 2^{15} i$$