HW 8 Solutions
21, 22, 26, 29, 32,
33, 35

Problem 15

Show

$$\mathcal{L}\left[\int_0^t \frac{f(u)}{u} du\right] = \frac{1}{s}\int_s^\infty F(v)\, dv$$

Given

Property of eqn (16): $\mathcal{L}\left[\int_0^t f(u)\, du\right] = \frac{1}{s} F(s)$

Result of Problem 13: $\int_s^\infty F(v)\, dv = \mathcal{L}\left[f\frac{(t)}{t}\right]$

Answer:

Let

$$g(t) = \frac{f(t)}{t}$$

Therefore,

$$\mathcal{L}\left[\int_0^t g(u)\, du\right] = \frac{1}{s} G(s)$$

But

$$\mathcal{L}\,[g(t)] = G(s) = \mathcal{L}\left[f\frac{(t)}{t}\right] = \int_s^\infty F(v)\, dv$$

Therefore,

$$\mathcal{L}\left[\int_0^t \frac{f(u)}{u}\, du\right] = \frac{1}{s}\int_s^\infty F(v)\, dv$$

Problem 21

(a) Show that if

$$\frac{d}{ds}F(s) = \mathcal{L}\,[g(t)]$$

then

$$\mathcal{L}^{-1}\,[F(s)] = \frac{-g(t)}{t}$$

But by definition

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Therefore,

$$\frac{d}{ds}F(s) = -\int_0^\infty e^{-st} t\, f(t)\, dt$$ differentiation under integral sign is allowed.

Therefore,

$$g(t) = -tf(t) \text{ or } f(t) = \mathcal{L}^{-1}\,[F(s)] = \frac{-g(t)}{t}$$

b)

$$\mathcal{L}^{-1}\left[\cot^{-1} s\right] = \frac{\sin t}{t} \qquad \text{i.e. } g(t) = -\sin t$$

Therefore,

$$\begin{aligned}
\mathcal{L}\,[g(t)] &= \frac{-1}{s^2+1} = -\frac{d}{ds}F(s) \text{ therefore } F(s) = \cot^{-1} s \\
\text{i.e. } \mathcal{L}^{-1}\left[\cot^{-1} s\right] &= \frac{\sin t}{t} \\
\mathcal{L}^{-1}\left[\log\frac{(s+1)}{s}\right] &= \frac{1-e^{-t}}{t} \text{ therefore } g(t) = e^{-t} - 1 \\
\mathcal{L}\,[g(t)] &= \frac{1}{s+1} - \frac{1}{s} = \frac{d}{ds}F(s) \text{ therefore } F(s) = \log\frac{s+1}{s} \\
\text{i.e. } \mathcal{L}^{-1}\left[\log\frac{(s+1)}{s}\right] &= \frac{1-e^{-t}}{t}
\end{aligned}$$

Problem 22

a)

$$\mathcal{L}^{-1}\left[\log\frac{(s-b)}{s-a}\right] = \frac{e^{at}-e^{bt}}{t}$$

Therefore,

$$\begin{aligned} g(t) &= e^{bt} - e^{at} \\ \mathcal{L}\,[g(t)] &= \mathcal{L}\,\left[e^{bt}\right] - \mathcal{L}\,\left[e^{at}\right] = \frac{1}{s-b} - \frac{1}{s-a} = +\frac{d}{ds}F(s) \end{aligned}$$

Therefore,

$$F(s) = \log\frac{[s-b]}{[s-a]}$$

b)

$$\mathcal{L}^{-1}\left[\tanh^{-1} s/a\right] = \frac{\sinh at}{t}$$

Therefore

$$\begin{aligned} g(t) &= -\sinh at \\ \mathcal{L}\,[g(t)] &= -\frac{a}{s^2-a^2} = \frac{d}{ds}F(s) \text{ therefore } F(s) = \tanh^{-1} s/a \end{aligned}$$

Problem 29

$$\begin{aligned} y(t) &= f(t) + \int_0^t g(t-u)\, y(u)\, du \\ \mathcal{L}\,[f(t)] &= F(s) \qquad \mathcal{L}\,[g(t)] = G(s) \end{aligned}$$

a)

$$\begin{aligned} y(s) &= F(s) + \mathcal{L}\,\left[\int_0^t g(t-u)\, g(u)\, du\right] \\ &= F(s) + G(s)y(s) \\ y(s)\,[1-G(s)] &= F(s) \text{ or } y(s) = \frac{F(s)}{1-G(s)} \\ y(s) &= F(s) + \underbrace{\frac{G(s)}{1-G(s)}}^{H(s)} F(s) \end{aligned}$$

b)

$$\begin{aligned} y(t) &= f(t) + \mathcal{L}^{-1}\,[H(s)F(s)] \\ &= f(t) + \int_0^t h(t-u)\, f(u)\, du \end{aligned}$$

where $H(s) = \dfrac{G(s)}{1-G(s)}$

c) Illustrate result with $G(t) = e^{-at}$ $\quad G(s) = \dfrac{1}{s+a}$

$$H(s) = \frac{1}{s+a}\cdot\frac{1}{1-1/s+a} = \frac{1}{\cancel{s+a}}\,\frac{(\cancel{s+a})}{[s+a-1]} = \frac{1}{s+a-1}$$

Therefore,

$$h(t) = e^{-(a+1)\,t}$$

26. By definition, $f * g = \int_0^t f(t-u)\, g(u)\, du$

a. $\int_0^t 1 \cdot \sin(au)\, du = -\frac{\cos(au)}{a}\Big|_0^t = \frac{1-\cos(at)}{a}$

b. $\int_0^t (t-u)e^{au}\, du = \int_0^t t e^{au}\, du - \int_0^t u e^{au}\, du$

The first term is $\frac{t}{a}(e^{at}-1)$

For the second: Let $e^{au}du = dV \Rightarrow V = \frac{e^{au}}{a}$

$$\int_0^t u e^{au}\, du \to \int_0^t u\, dV = uV\Big|_{u=0}^t - \int_0^t V\, du$$
$$= \frac{t}{a}e^{at} - \frac{1}{a^2}(e^{at}-1)$$

Subtracting the second from the first,

$$-t/a + \frac{1}{a^2}(e^{at}-1) = \frac{1}{a^2}(e^{at} - at - 1)$$

c. $\int_0^t e^{a(t-u)} e^{bu}\, du = e^{at}\int_0^t e^{(b-a)u}\, du = e^{at}/(b-a) \cdot \left[e^{(b-a)t} - 1\right]$

$$= \frac{e^{bt} - e^{at}}{b-a}$$

d. $\int_0^t \sin(a(t-u))\sin(bu)\, du$

$$= -\frac{1}{4}\int_0^t \left(e^{ia(t-u)} - e^{-ia(t-u)}\right)\left(e^{ibu} - e^{-ibu}\right) du$$

$$= -\frac{1}{4}\int_0^t e^{i[at+(b-a)u]} - e^{i[-at+(b+a)u]} - e^{i[at+(-a-b)u]} + e^{i[-at+(a-b)u]}\, du$$

$$= \frac{-1}{4i}\left[\frac{e^{ibt} - e^{iat}}{b-a} - \frac{e^{ibt} - e^{-iat}}{b+a} + \frac{e^{-ibt} - e^{iat}}{b+a} - \frac{e^{-ibt} - e^{-iat}}{b-a}\right]$$

$$= \frac{-1}{4i}\left[\frac{1}{b-a}\left(e^{ibt} - e^{-ibt} - e^{iat} + e^{-iat}\right) + \frac{1}{b+a}\left(-e^{ibt} + e^{-ibt} - e^{iat} + e^{-iat}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin(at) - \sin(bt)}{b-a} + \frac{\sin(bt) + \sin(at)}{b+a}\right]$$

$$= \left[(b+a)(\sin at - \sin bt) + (b-a)(\sin bt + \sin at)\right] / 2(b^2-a^2)$$

$$= \frac{b\sin at - a\sin bt}{b^2 - a^2}$$

Chapter 2 HW Solutions

29. $y(t) = f(t) + \int_0^t G(t-u)\,y(u)\,du$

$Y(s) = F(s) + G(s)Y(s)$

a) $Y(s) = \dfrac{F(s)}{1-G(s)} = F(s) + \underbrace{\dfrac{G(s)}{1-G(s)}}_{H(s)} F(s)$

b) $\therefore\ y(t) = f(t) + \int_0^t h(t-u) f(u)\,du$

c) $G(t) = e^{-at}$ $\quad G(s) = \frac{1}{s+a}$ $\quad H(s) = \frac{1}{s+(a-1)}$

$\therefore\ y(t) = f(t) + \int_0^t e^{-(a-1)(t-u)} f(u)\,du$

32. a) $H(t) = 1 \quad t>0$

$H(t) = 0 \quad t<0$

$\mathcal{L}\{H(t)\} = \int_0^\infty e^{-st}dt = 1/s \Rightarrow \mathcal{L}\{H(t-t_1)\} = e^{-st_1}/s$

b) $\mathcal{L}\{\delta(t)\} = 1 \quad \mathcal{L}\{\delta'(t)\} = s\ ,\ \mathcal{L}\{H(t)\} = \frac{1}{s}$

$\therefore\ \mathcal{L}\{\delta'(t)\} = s\,\mathcal{L}\{\delta(t)\} = s^2\,\mathcal{L}\{H(t)\}$

Derivation $\mathcal{L}\{\frac{d^2y}{dt^2}\} = s\,\mathcal{L}\{\frac{dy}{dt}\} = s^2\,\mathcal{L}\{y(t)\}$

- ignoring initial condition.

33 a) $\mathcal{L}^{-1}\{\frac{1}{s^2-3s+2}\} = \mathcal{L}^{-1}\{\frac{1}{s-2}\} - \mathcal{L}^{-1}\{\frac{1}{s-1}\} = e^{2t} - e^{t}$

b) $\mathcal{L}^{-1}\{\frac{s}{s^2-2s+5}\} = \mathcal{L}^{-1}\{\frac{s}{(s-1)^2+2^2}\}$ $\qquad a=1 \quad b=2$

$H(s) = s \qquad H(1+2i) = 1+2i \qquad \alpha_r = 1 \quad \alpha_i = 2$

$\therefore\ \mathcal{L}^{-1}\{\frac{s}{s^2-2s+5}\} = \frac{1}{2}\{2\cos 2t + \sin 2t\}\,e^{t}$

c) $\mathcal{L}^{-1}\{\frac{s+1}{s^4+1}\} = \mathcal{L}^{-1}\{\frac{s}{s^4+1}\} + \mathcal{L}^{-1}\{\frac{1}{s^4+1}\}$

$= \mathcal{L}^{-1}\{\frac{s}{s^4+4(1/\sqrt{2})^4}\}_{T\,28} + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\{\frac{4(1/\sqrt{2})^3}{s^4+4(1/\sqrt{2})}\}_{T\,27}$ $\qquad a = \frac{1}{\sqrt{2}}$

$= \sin(t/\sqrt{2})\sinh(t/\sqrt{2}) + \frac{1}{\sqrt{2}}\left(\sin(t/\sqrt{2})\cosh(t/\sqrt{2}) - \cos(t/\sqrt{2})\sinh(t(\sqrt{2}))\right)$

d) $\mathcal{L}^{-1}\{\frac{2s+1}{s(s+1)(s+2)}\}$ $\qquad$ linear factors $(s-a)$ with $a = 0,-1,-2$

$a=0 \quad H(s) = \frac{2s+1}{(s+1)(s+2)} \quad H(0) = \frac{1}{2} \qquad \frac{1}{2}e^0 = 1/2$

$a=-1 \quad H(s) = \frac{2s+1}{s(s+2)} \quad H(-1) = \frac{-1}{-1} = 1 \Rightarrow e^{-t}$

$a=-2 \quad H(s) = \frac{2s+1}{s(s+1)},\ H(-2) = -3/2 \Rightarrow -3/2\,e^{-2t}$

$\mathcal{L}^{-1}\{\frac{2s+1}{s(s+1)(s+2)}\} = 1/2\,(1 + 2e^{-t} - 3e^{-2t})$

e) $\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\} - \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}$ $\qquad a=0 \ \ b=1 \quad \alpha_r = 1 \ \ \alpha_i = 0$

$= t - \sin t$

f) $\mathcal{L}^{-1}\{\frac{e^{-s}}{s+1}\} = \mathcal{L}^{-1}\{e^{-s}F(s)\} = u(t-1)f(t)$ with $f(t) = e^{-t}$

$\mathcal{L}^{-1}\{\frac{e^{-s}}{s+1}\} = \begin{cases} e^{-(t-1)} & t>1 \\ 0 & 0<t<1 \end{cases}$

35. a) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\}$ $\quad a=-1,\ b=1$

$H(s) = s+1 \quad H(-1+i) = i \quad \alpha_r = 0 \quad \alpha_i = 1$

$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+2}\right\} = \cos t\, e^{-t}$

b) $\mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\} = \mathcal{L}^{-1}\{s F(s)\} \quad F(s) = \frac{s}{s^4+4a^4}$ T 28

$\frac{1}{2a^2}\mathcal{L}^{-1}\left\{\frac{2a^2 s}{s^4+4a^4}\right\} = \frac{1}{2a^2}(\sin at \ \sinh at)$

$\therefore \mathcal{L}^{-1}\{s F(s)\} = \frac{1}{2a^2}\frac{d}{dt}(\sin at \sinh at) = \frac{1}{2a}[\cos at \sinh at + \sin at \cosh at]$

c) $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \mathcal{L}^{-1}\{e^{-\pi s} F(s)\} \quad f(t) = \sin t$

$\cancel{e^{-\pi s}F(s) = \mathcal{L}\{f(t-\pi)\}} \cdot e^{-\pi s} F(s) = \mathcal{L}\{u(t-\pi) f(t-\pi)\}$

$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \begin{cases} \sin(t-\pi) & t > \pi \\ 0 & 0 < t < \pi \end{cases}$

d) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+a)^3}\right\} = \mathcal{L}^{-1}\{s F(s)\} \quad F(s) = \frac{s}{(s+a)^3}$ T37 $n=3$

$f(t) = \frac{(2-at)}{2}\, t\, e^{-at}$

$\mathcal{L}^{-1}\left\{\frac{s^2}{(s+a)^3}\right\} = \frac{d}{dt} f(t) = \frac{1}{2}(2-4at+a^2t^2)e^{-at}$

e) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\}$ T38 $n=3$ $a=1$ $= \frac{1}{8}\left[\sqrt{\frac{\pi}{2}}\, t^{5/2} J_{5/2}(t)\right]$

$J_{5/2}(t) = \frac{3}{t} J_{3/2}(t) - J_{1/2}(t) = \frac{3}{t}\left(\frac{1}{t}J_{1/2}(t) - J_{-1/2}(t)\right) - J_{1/2}(t)$

$= \frac{(3-t^2)}{t^2} J_{1/2}(t) - \frac{3}{t} J_{-1/2}(t)$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\} = \frac{1}{8}\left[\sqrt{\frac{\pi}{2}}\, t^{5/2}\left[\frac{(3-t^2)}{t^2}\cdot\sqrt{\frac{2}{\pi}}\sin t - \frac{3}{t}\sqrt{\frac{2}{\pi}}\cos t\right]\right]$

$= \frac{1}{8}\left[(3-t^2)\sin t - 3t\cos t\right]$

35. f) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)^3}\right\}$ T40 with $a=1$ $n=3$

$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)^3}\right\} = \frac{1}{8}\sqrt{\frac{\pi}{2}}\, t^{5/2} I_{5/2}(t) = \frac{1}{8}\sqrt{\frac{\pi}{2}}\, t^{5/2}\left[\left(\frac{t^2+3}{t^2}\right) I_{1/2} - \frac{3}{t} I_{-1/2}(t)\right]$

$= \frac{1}{8}\sqrt{\frac{\pi}{2}}\, t^{5/2}\left[\frac{(t^2+3)}{t^2}\sqrt{\frac{2}{\pi t}}\sinh t - \frac{3}{t}\sqrt{\frac{2}{\pi t}}\cosh t\right]$

$= \frac{1}{8}\left[(3+t^2)\sinh t - 3t\cosh t\right]$

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40 a) $sY(s) - 1 + kY(s) = 0 \quad Y(s) = \frac{1}{s+k} \Leftrightarrow y(t) = e^{-kt}$

b) $sY(s) + kY(s) = 1/s \quad Y(s) = \frac{1}{s(s+k)} \Leftrightarrow y(t) = \frac{(1-e^{-kt})}{k}$

c) $sY(s) - 1 + kY(s) = e^{-s} \quad Y(s) = \frac{1}{s+k} + \frac{e^{-s}}{s+k}$

$y(t) = \begin{cases} e^{-kt} + e^{-k(t-1)} & t > 1 \\ e^{-kt} & 0 < t < 1 \end{cases}$

d) $sY(s) - y_0 + kY(s) = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$\therefore Y(s) = \frac{y_0}{s+k} + \frac{F(s)}{s+k}; \quad y(t) = e^{-kt}\left[y_0 + \int_0^t e^{k\alpha} f(\alpha)\,d\alpha\right]$

43. $(s^2+1)Y(s) - C = e^{-as} \qquad C = \frac{dy(0)}{dt}$

$Y(s) = \frac{C}{s^2+1} + \frac{e^{-as}}{s^2+1} \Longleftrightarrow y(t) = C\sin t + \sin(t-a)$

$y(b) = 0 = C\sin b + \sin(b-a) \Rightarrow C = \frac{\sin(a-b)}{\sin b}$

$\therefore y(t) = \begin{cases} \frac{\sin(a-b)}{\sin b}\sin t + \sin(t-a) & a \le t \le b \\ \frac{\sin(a-b)}{\sin b}\sin t & 0 \le t \le a \end{cases}$