HW 9
Spring 09

Key

| problem | worth |
|---|---|
| 1 | 2 |
| 2 | 1 |
| 5 | 1 |
| 6 | 1 |
| 22 | 2 |
| 23 | 1 |
| 26 | 1 |
| 28 | 1 |
| total | 10 |

5.1

1) Show That

$$\frac{d^2y}{dx^2} - k^2 y = 0 \qquad y(0) = y(\ell) = 0$$

cannot have a nontrivial solution for real values of k.

General soln:

$$y(x) = C_1 e^{kx} + C_2 e^{-kx}$$

Boundary conditions

$$y(0) = C_1 + C_2 = 0$$

$$y(l) = C_1 e^{k\ell} + C_2 e^{-k\ell} = 0$$

Determinant must equal zero for a nontrivial solution

$$\begin{vmatrix} 1 & 1 \\ e^{k\ell} & e^{-k\ell} \end{vmatrix} = -2 \sinh(k\ell) \neq 0 \quad \text{for real } k.$$

$\therefore$ only solution is the trivial one $y(x) = 0$.

5.1

2.) Determine the values of k for which the partial differen[tial]

equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

possesses nontrivial solutions of the form $T(x,y) = f(x) \sinh ky$.

which vanish when $x=0$ and $x=l$

$$\frac{\partial^2 T}{\partial x^2} = \sinh ky \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 T}{\partial y^2} = k^2 \sinh ky\, f(x)$$

$$\rightarrow \left[\frac{d^2 f}{dx^2} + k^2 f(x)\right] \sinh ky = 0$$

$$\Rightarrow \frac{d^2 f}{dx^2} + k^2 f(x) = 0 \quad \text{with } f(0) = f(l) = 0.$$

General soln.

$$f(x) = C_1 \cos kx + C_2 \sin kx$$

$$f(0) = C_1 = 0$$

$$f(l) = C_2 \sin kl = 0 \qquad C_2 \neq 0 \text{ for nontrivial solution}$$

$$\Rightarrow k = \frac{n\pi}{l} \qquad n = 1, 2, 3, \ldots$$

5.2 5

5) Uniform string unrestrained from transverse motion at $x=0$ and attached to yielding support at $x=l$ obeys

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

with boundary conditions

at $x=0$ $\qquad y'(0)=0$ $\qquad$ free end condition

$x=l$ $\qquad \alpha l y'(l) = -y(l)$ $\qquad$ yielding support with spring constant $k$.

$\alpha = \frac{T}{kl}$ ← tension

$y'(0) = 0 \quad \rightarrow \quad y(x) = c \cos\sqrt{\lambda}x$

$$y'(x) = -c\sqrt{\lambda}\sin\sqrt{\lambda}x$$

$$+\alpha l \cancel{c}\sqrt{\lambda}\sin\sqrt{\lambda}l = +\cancel{c}\cos\sqrt{\lambda}l$$

Let $\mu = \sqrt{\lambda}l$.

$$\alpha\mu_n = \cot\mu_n$$

, Trancendental equation which yields the roots $\mu_n$.

From Sec. 5.2 $\quad \omega = \sqrt{\frac{\lambda T}{\rho}}$ $\quad$ and $\lambda = \mu^2/l^2$

$$= \frac{\mu}{l}\sqrt{T/\rho}$$

∴ n-th critical speed is $\omega_n = \frac{\mu_n}{l}\sqrt{\frac{T}{\rho}}$

where $\mu_n$ is a root of transcendental eqn. above.

(6) Uniform rotating string attached to yielding supports of modulus $k_1$ at $x=0$ and $k_2$ at $x=l$.

i.e. Boundary conditions on equation in (5) are

$$\alpha_1 l\, y'(0) = y(0) \qquad \alpha_2 l\, y'(l) = -y(l)$$

where $\alpha_1 = \dfrac{T}{k_1 l}$, $\alpha_2 = \dfrac{T}{k_2 l}$.

General soln: $\quad y(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$\therefore\ y'(0) = B\sqrt{\lambda}$

$y'(l) = -A\sqrt{\lambda}\sin\sqrt{\lambda}\,l + B\sqrt{\lambda}\cos\sqrt{\lambda}\,l$

$\therefore\ y(x) = B\left(\sin\mu_n x/l + \alpha_1\mu_n\cos\mu_n x/l\right)$ n-th deflection mode.

at $x=0$: $\quad \alpha_1 l B\sqrt{\lambda} = A \Rightarrow B = \dfrac{A}{\alpha_1 l\sqrt{\lambda}} = \dfrac{A}{\mu\alpha_1}$

at $x=l$. $\quad \alpha_2 l\left(-A\sqrt{\lambda}\sin\sqrt{\lambda}\,l + B\sqrt{\lambda}\cos\sqrt{\lambda}\,l\right) = -A\cos\sqrt{\lambda}\,l - \dfrac{A}{\alpha_1 l\sqrt{\lambda}}\sin\sqrt{\lambda}\,l$

$$\left(\frac{1}{\alpha_1 l\sqrt{\lambda}} - \alpha_2 l\sqrt{\lambda}\right)\sin\sqrt{\lambda}\,l = -\left(\frac{\alpha_2}{\alpha_1} + 1\right)\cos\sqrt{\lambda}\,l$$

$$(1 - \alpha_1\alpha_2\mu_n^2)\tan\mu_n = -(\alpha_1 + \alpha_2)\mu_n$$

But $\omega_n = \dfrac{\mu_n}{l}\sqrt{\dfrac{T}{\rho}}$ is n-th critical speed. from problem

5.6.

22 (a) $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + (x+\lambda)y = 0$

$a_0(x) = x$, $a_1(x) = 2$ $a_2(x) = x$, $a_3(x) = 1$

$p = e^{\int \frac{2}{x}dx} = x^2$ $q = \dfrac{x}{x}.x^2 = x^2$, $r(x) = \dfrac{1}{x}.x^2 = x$

$\rightarrow (x^2y')' + [x^2 + \lambda x]y$

b) $\dfrac{d^2y}{dx^2} + \cot x \dfrac{dy}{dx} + \lambda y = 0$

$a_0(x) = 1$, $a_1(x) = \cot x$ $a_2(x) = 0$. $a_3(x) = 1$

$p = e^{\int \cot x . dx} = \sin x$ $q = 0$ $r(x) = 1.p$

$\rightarrow (\sin x\, y')' + \lambda y \sin x = 0$

c) $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + (b+\lambda)y = 0$ a, b consts

$a_0(x) = 1$ $a_1(x) = a$ $a_2(x) = b$ $a_3(x) = 1$

$p = e^{a\int dx} = e^{ax}$ $q = b e^{ax}$ $r(x) = e^{ax}$

$\rightarrow (e^{ax}y')' + (be^{ax} + \lambda e^{ax})y = 0$

22 d)

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay + \lambda y = 0 \qquad a, c \text{ constants}$$

$$a_0(x) = x \quad a_1(x) = c-x \quad a_2(x) = -a \quad a_3(x) = 1$$

$$p = e^{\int \frac{c-x}{x}dx} = e^{c\ln x}\, e^{-x} = x^c e^{-x}$$

$$q = -\frac{a}{x}\cdot x^c e^{-x} = -a x^{c-1} e^{-x} \qquad r(x) = \frac{1}{x}\cdot x^c e^{-x} = x^{c-1} e^{-x}$$

$$\rightarrow \left(x^c e^{-x} y'\right)' + \left[-a x^{c-1} e^{-x} + x^{c-1} e^{-x} \lambda\right] y = 0.$$

23)

$$\frac{d^2y}{dx^2} + \mu^2 y = 0 \quad , \quad y(0) = 0 \, , \; \alpha l y'(l) + y(l) = 0$$

$$\alpha > 0$$

general solution $\quad y(x) = C \sin \mu x$

B.C. $x = l$.

$$C \alpha l \mu \cos \mu l + C \sin \mu l = 0 \qquad (*)$$

$$\therefore \; \alpha l \mu + \tan \mu l = 0$$

$\mu_n$ are roots of this transcendental equation with eigenfunctions $\varphi_n(x) = C \sin \mu_n x$.

Corresponds to Sturm-Liouville problem with $p = 1, q = 0$ and $r = 1$

$\therefore$ Orthogonality condition: $\displaystyle\int_0^l \varphi_n(x)\varphi_m(x)\,dx = \int_0^l \sin\mu_n x \sin\mu_m x\,dx$

Must show that $\; p(x)\left[\varphi_n(x)\varphi_m'(x) - \varphi_m(x)\varphi_n'(x)\right]_0^l = 0$

with $p = 1$ this gives

$$\sin(\mu_n l)\underbrace{\mu_m \cos(\mu_m l)}_{\frac{1}{\alpha l}\sin(\mu_m l)} - \sin(\mu_m l)\underbrace{\mu_n \cos(\mu_n l)}_{\frac{1}{\alpha l}\sin(\mu_n l)} = 0$$

from $(*)$

$\therefore$ eigenfunctions are orthogonal.

26)

$$(py')' + (q + \lambda r)y = 0$$

$$\left\{ p(x) \left[ \varphi_n(x)\varphi_m'(x) - \varphi_m(x)\varphi_n'(x) \right] \right\}_a^b$$

$$p(b)\left[\varphi_n(b)\varphi_m'(b) - \varphi_m(b)\varphi_n'(b)\right]$$
$$- p(a)\left[\varphi_n(a)\varphi_m'(a) - \varphi_m(a)\varphi_n'(a)\right]$$

$p(a) = p(b)$

$$\left[\varphi_n(b)\varphi_m'(b) - \varphi_n(a)\varphi_m'(a) - \varphi_m(b)\varphi_n'(b) + \varphi_m(a)\varphi_n'(a)\right] = 0$$

$\downarrow$

$$\Big[(\alpha_{11}\varphi_n(a) + \alpha_{12}\varphi_n'(a))\ (\alpha_{21}\varphi_m(a) + \alpha_{22}\varphi_m'(a))$$
$$- (\alpha_{11}\varphi_m(a) + \alpha_{12}\varphi_m'(a))(\alpha_{21}\varphi_n(a) + \alpha_{22}\varphi_n'(a)) - \varphi_n(a)\varphi_m'(a)$$
$$+ \varphi_m(a)\varphi_n'(a) = 0$$

$$= \Big\{ \alpha_{11}\alpha_{21}\cancel{\varphi_n(a)\varphi_m(a)} + \alpha_{11}\alpha_{22}\varphi_n(a)\varphi_m'(a) + \alpha_{12}\alpha_{21}\varphi_n'(a)\varphi_m(a)$$
$$+ \cancel{\alpha_{12}\alpha_{22}\varphi_n'(a)\varphi_m'(a)} - \cancel{\alpha_{11}\alpha_{21}\varphi_m(a)\varphi_n(a)} - \alpha_{11}\alpha_{22}\varphi_m(a)\varphi_n'(a)$$
$$- \alpha_{12}\alpha_{21}\varphi_m'(a)\varphi_n(a) - \cancel{\alpha_{12}\alpha_{22}\varphi_m'(a)\varphi_n'(a)} - \varphi_n(a)\varphi_m'(a)$$
$$+ \varphi_m(a)\varphi_n'(a) \Big\} = 0$$

$$= \Big\{ \alpha_{11}\alpha_{22}\varphi_n(a)\varphi_m'(a) + \alpha_{12}\alpha_{21}\varphi_n'(a)\varphi_m(a) - \alpha_{12}\alpha_{21}\varphi_m'(a)\varphi_n(a)$$
$$- \alpha_{11}\alpha_{22}\varphi_m(a)\varphi_n'(a) - \varphi_n(a)\varphi_m'(a) + \varphi_m(a)\varphi_n'(a) \Big\} = 0$$

16) Cont

$$(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\varphi_n(a)\,\varphi_m'(a) - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\varphi_n'(a)\,\varphi_m(a)$$

$$- \varphi_n(a)\,\varphi_m'(a) + \varphi_n'(a)\,\varphi_m(a) = 0$$

$$[\underbrace{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} - 1}]\,\varphi_n(a)\,\varphi_m'(a) - [\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} - 1]\,\varphi_n'(a)\,\varphi_m(a)$$

$$= 0$$

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = 1 \quad \rightarrow \quad \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1.$$

28)

## Sturm – Liouville Problem

$$(py')' + (q + \lambda r)y = 0 \qquad (1)$$

Prove that eigenvalues $\lambda$ are real! $p, q, r$ are real and $r > 0$

Assume contrary! ie $\lambda$ and $\bar{\lambda}$ are distinct eigenvalues satisfying (1) with eigenfunctions $\varphi(x)$ and $\bar{\varphi}(x)$

i.e $(p\varphi')' + (q + \lambda r)\varphi = 0$

$(p\bar{\varphi}')' + (q + \bar{\lambda} r)\bar{\varphi} = 0$

from orthogonality property $\quad \varphi(x) = u(x) + i\,v(x)$

$$(\lambda - \bar{\lambda}) \int_a^b r\varphi\bar{\varphi}\,dx = (\lambda - \bar{\lambda}) \int_a^b r(u^2 + v^2)\,dx = 0$$

But since $r > 0$, the integral cannot be zero (for nontrivial soln)

$\therefore \quad \lambda = \bar{\lambda} \quad \Rightarrow \quad \lambda$ real.