Homework #7

Solutions

11)

a) $2xy'' + (1-2x)y' - y = 0$

$$y = \sum_{k=0}^{\infty} A_k x^{s+k} \qquad y' = \sum_{k=0}^{\infty} (s+k) A_k x^{s+k-1}$$

$$y'' = \sum_{k=0}^{\infty} (s+k)(s+k-1) A_k x^{s+k-2}$$

$$2\sum_{k=0}^{\infty}(s+k) A_k x^{s+k-1}(s+k-1) + \sum_{k=0}^{\infty}(s+k)A_k\left[x^{s+k-1} - 2x^{s+k}\right] - \sum_{k=0}^{\infty} A_k x^{s+k} = 0$$

$$= 2\sum_{k=-1}^{\infty}(s+k+1)A_{k+1}x^{s+k}(s+k) + \sum_{k=-1}^{\infty}(s+k+1)A_{k+1}x^{s+k}$$

$$-2\sum_{k=0}^{\infty}A_k(s+k)x^{s+k} - \sum_{k=0}^{\infty}A_k x^{s+k} = 0$$

Indicial equation

$$2s(s-1) + s = 0$$

or $\quad 2s^2 - s = 0 \quad \Rightarrow \quad s = 0, \tfrac{1}{2}$

$s=0$

$$\Rightarrow \sum_{k=0}^{\infty}\left[2(k+1)A_{k+1}(k)x^k + (k+1)A_{k+1}x^k - (2k+1)A_k\right]x^{s+k} = 0$$

$$\therefore \quad (2k+1)(k+1)A_{k+1} = (2k+1)A_k$$

$$\Rightarrow \quad A_{k+1} = \frac{A_k}{k+1}; \quad k \neq -\tfrac{1}{2}$$

$\Rightarrow \quad y_1 = A_0 + \frac{A_0}{1}x + \frac{A_1}{2}x^2 + \frac{A_2}{3}x^3 + \cdots$

But $\quad A_1 = A_0, \quad A_2 = \frac{A_1}{2} = \frac{A_0}{2!}$ & so on

$$\Rightarrow \quad y_1 = A_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = A_0 e^x$$

$s = 1/2$

$$\Rightarrow \sum_{k=0}^{\infty}\left[(2k+2)\left(k+\frac{3}{2}\right)A_{k+1} - (2k+2)A_k\right]x^{k+1/2} = 0$$

$$\Rightarrow \quad A_{k+1} = \frac{2(k+1)A_k}{2(k+1)(k+3/2)} = \frac{A_k}{k+3/2}$$

$\therefore \quad A_1 = \frac{A_0}{3/2} = \frac{2A_0}{3}, \quad A_2 = \frac{A_1}{5/2} = \frac{4A_0}{(3)(5)},$

$A_3 = \frac{A_2}{7/2} = \frac{2^3 A_0}{(3)(5)(7)}$ & so on

$$\therefore \quad y_2 = A_0\left(1 + \frac{2x}{3} + \frac{2^2x^2}{1\cdot3\cdot5} + \frac{2^3x^3}{1\cdot3\cdot5\cdot7} + \cdots\right)$$

$$\therefore \quad y = c_1 y_1 + c_2 x^{1/2} y_2$$

$$\Rightarrow \quad \boxed{y = c_1 e^x + c_2 x^{1/2}\left(1 + \frac{2x}{1\cdot3} + \frac{(2x)^2}{1\cdot3\cdot5} + \frac{(2x)^3}{1\cdot3\cdot5\cdot7} + \cdots\right)}$$

b) $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

$$y = \sum_{k=0}^{\infty} A_k x^{s+k} \; ; \quad y' = \sum_{k=0}^{\infty} (s+k) A_k x^{s+k-1} \; ; \quad y'' = \sum_{k=0}^{\infty} (s+k)(s+k-1) x^{s+k-2}$$

$$\sum_{k=0}^{\infty} A_k (s+k)(s+k-1) x^{s+k} + \sum_{k=0}^{\infty} (s+k) A_k x^{s+k}$$
$$+ \sum_{k=0}^{\infty} A_k \left(x^{s+k+2} - \frac{1}{4} x^{s+k} \right) = 0$$

$$= \left\{ \left[A_0 (s-1) s + s A_0 - \frac{1}{4} A_0 \right] + (s+1) s A_1 x + (s+1) A_1 x \right.$$
$$\left. - \frac{1}{4} A_1 x + \sum_{k=2}^{\infty} \left[(s+k)^2 A_k + A_{k-2} - \frac{1}{4} A_k \right] x^k \right\} x^s = 0$$

$\therefore$ Indicial equation

$$(s-1)s + s - \frac{1}{4} = 0 \Rightarrow s^2 - \frac{1}{4} = 0$$

$$s = \pm 1/2$$

$s = 1/2$

$$\Rightarrow \left(k + \frac{1}{2}\right)^2 A_k + A_{k-2} - \frac{1}{4} A_k = 0$$

$$\Rightarrow A_k = \frac{-A_{k-2}}{\left(k + \frac{1}{2}\right)^2 - \frac{1}{4}}$$

Since A_{-1} is undefined, A_1 is arbitrary $\{ A_1 = 0 \text{ if } A_{-1} = 0 \}$

$\therefore A_0, A_1, A_2 = \frac{-A_0}{3 \cdot 2}, A_3 = \frac{-A_1}{4 \cdot 3}$ and so on

$\therefore$

$$y_1 = A_0 x^{1/2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right]$$

$s = -1/2$

$$A_k = -\frac{A_{k-2}}{(k-\frac{1}{2})^2 - \frac{1}{4}}$$

A_0 is arbitrary

$A_1 = 0$, $A_3 = 0$ & so on

$$A_2 = -\frac{A_0}{2}, \quad A_4 = -\frac{A_2}{12} = -\frac{A_0}{24} \quad \text{\& so on}$$

$$\therefore y_2 = A_0 x^{-1/2}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right]$$

$$\therefore \quad y = c_1 y_1 + c_2 y_2$$

$$\Rightarrow \boxed{y = c_1 x^{-1/2}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right] + c_2 x^{1/2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - + \cdots\right]}$$

(c) $xy'' + 2y' + xy = 0$

$$y = \sum_{k=0}^{\infty} A_k x^{s+k}; \quad y' = \sum_{k=0}^{\infty} (s+k) A_k x^{s+k-1}; \quad y'' = \sum_{k=0}^{\infty} (s+k)(s+k-1) A_k x^{s+k-2}$$

$$\Rightarrow \sum_{k=0}^{\infty} (s+k)(s+k-1) A_k x^{s+k-1} + 2\sum_{k=0}^{\infty} (s+k) A_k x^{s+k-1} + \sum_{k=0}^{\infty} A_k x^{s+k+1} = 0$$

$\therefore$ Indicial equation

$$s(s-1)A_0 + 2sA_0 = 0$$

$$\Rightarrow (s^2+s) = 0 \Rightarrow \quad s = 0, -1$$

$s = 0$

$$\Rightarrow x^s \left[\sum_{k=0}^{\infty} \{ [s(s+1) + 2s] A_k + A_{k-2} \} x^{k} \right] = 0$$

$$\Rightarrow A_k = \frac{-A_{k-2}}{k(k+1)}$$

$\therefore$ $A_1 = 0$

A_0 is arbitrary

$$A_2 = \frac{-A_0}{2 \cdot 3}, \quad A_4 = \frac{-A_2}{4(5)} = \frac{+A_0}{2 \cdot 3 \cdot 4 \cdot 5}$$

and so on

$$\therefore y_1 = A_0 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - + \cdots \right)$$

$s = -1$

$$x^s \left[\sum_{k=0}^{\infty} x^k \{ (k-1)(k-2) A_k + 2(k-1) A_k + A_{k-2} \} x^{-1} \right] = 0$$

$$\Rightarrow A_k = \frac{-A_{k-2}}{k(k-1)} \Rightarrow$$

A_0 arbitrary

$A_1 = 0$

$$A_2 = \frac{-A_0}{2}, \quad A_4 = \frac{-A_2}{4 \cdot 3} = \frac{A_0}{4!}, \text{ \& so on}$$

$$y_2 = A_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right]x^{-1}$$

$$\boxed{y = c_1 x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + c_2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)}$$

d) $x(1-x)y'' - 2y' + 2y = 0$

$$y = \sum_{k=0}^{\infty} A_k x^{s+k}$$

As before,

$$x^s\left[\sum_{k=0}^{\infty}(k+s)(s+k-1)A_k x^{k-1} - \sum_{k=0}^{\infty}(s+k)(s+k-1)A_k x^k - 2\sum_{k=0}^{\infty}A_k(s+k)x^{k-1} + 2\sum_{k=0}^{\infty}A_k x^k\right] = 0$$

$\therefore$ Indicial equation

$$s(s-1)A_0 - 2sA_0 = 0$$

$$\Rightarrow \quad s(s-3) = 0 \quad \Rightarrow \quad s = 0, 3$$

$s = 0$

$$\Rightarrow \sum_{k=1}^{\infty}\left\{k(k-3)A_k - [(k-1)(k-2) - 2]A_{k-1}\right\} = 0$$

$$\Rightarrow \quad A_k = \frac{[(k-1)(k-2) - 2]A_{k-1}}{k(k-3)}$$

$A_0, \quad A_1 = A_0, \quad A_2 = A_1 = A_0,$

no solution for A_3, A_4 etc

$\underline{y_1 = A_0(1 + x + x^2)}$

$\underline{s = 3}$

$$A_k = \frac{A_{k-1}\left[(k+2)(k+1) - 2\right]}{(k+3)\,k}$$

$\therefore \quad A_1 = \dfrac{A_0(4)}{4} = A_0, \qquad A_2 = \dfrac{A_1(10)}{10} = A_1 = A_0$

& so on

$\therefore \quad \underline{y_1 = A_0 x^3(1 + x + x^2 + x^3 + \cdots)}$

$$\therefore \quad y = c_1(1 + x + x^2) + c_2 x^3(1 + x + x^2 + \cdots)$$

$$= \underline{\underline{c_1(1 + x + x^2) + \frac{c_2 x^3}{1 - x}}} \quad \text{for } \underline{\underline{|x| < 1}}$$

12)

a) $x^2y'' - 2xy' + (2-x^2)y = 0$

$$y = \sum_{k=0}^{\infty} A_k x^{s+k}$$

$$\Rightarrow \quad x^s \left[\sum_{k=0}^{\infty} (s+k)(s+k-1)A_k x^k - 2\sum_{k=0}^{\infty}(s+k)A_k x^k + \sum_{k=0}^{\infty} A_k\left(2x^k - x^{k+2}\right)\right] = 0$$

$\therefore$ Indicial equation

$$s(s-1) - 2s + 2 = 0$$

$$\Rightarrow \quad s^2 - 3s + 2 = 0 \quad \Rightarrow \quad s^2 - s - 2s + 2 = 0$$

$$\Rightarrow \quad \underline{s = 1, 2}$$

We have the recursive equation,

$$A_k = \frac{A_{k-2}}{(s+k)(s+k-3)+2}$$

$\underline{s = 1}$

$$A_k = \frac{A_{k-2}}{(k+1)(k-2)+2} = \frac{A_{k-2}}{k(k-1)}$$

$\therefore \quad A_0, A_1, A_2 = \frac{A_0}{2}, A_3 = \frac{\cancel{A_1}}{6}, A_4 = \frac{A_2}{12} = \frac{A_0}{24}$

A_0, A_1 not defined

& so on

$$\therefore y_1 = A_0 x^{\bullet}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$S = 2$

$$A_k = \frac{A_{k-2}}{(k+2)(k-1)+2} = \frac{A_{k-2}}{k(k+1)}$$

$$\therefore A_0,\quad A_1 = 0,\quad A_2 = \frac{A_0}{3!},\quad A_4 = \frac{A_0}{5!}$$ & so on

$$\therefore y_2 = A_0\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)x^2$$

$$\therefore \boxed{y = C_1 x^{\bullet}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + C_2 x^2\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)}$$

(b) $(x-1)y'' - xy' + y = 0$

$$y = \sum_{k=0}^{\infty} A_k x^{s+k}$$

$$\Rightarrow \left[\sum_{k=0}^{\infty}(s+k)(s+k-1)A_k x^{s+k-1} - \sum_{k=0}^{\infty}(s+k)(s+k-1)A_k x^{s+k-2}\right.$$
$$\left. - \sum_{k=0}^{\infty}(s+k)A_k x^{s+k} + \sum_{k=0}^{\infty}A_k x^{s+k}\right] = 0$$

$\therefore$ Indicial equation

$$A_0 s(s-1) = 0 \Rightarrow s = 0, 1$$

Also, recursive equation,

$$A_k = \frac{(s+k-1)(s+k-2)A_{k-1} - (s+k-1)A_{k-2}}{(s+k)(s+k-1)}$$

$s = 1$

$$A_k = \frac{k(k-1)A_{k-1} - kA_{k-2}}{(k+1)k}$$

A_0, $A_1 = 0$,

$$A_2 = \frac{-2A_0}{3 \cdot 2} = -\frac{A_0}{3}, \quad A_3 = \frac{6A_2 - \cancel{3A_1}^{\,0}}{12} = -\frac{A_0}{6}$$

& so on

$$\therefore y_1 = -A_0\left(1 + \frac{x^2}{3} + \frac{x^3}{6} + \cdots\right)$$

$s = 0$

$$A_k = \frac{(k-1)(k-2)A_{k-1} - (k-1)A_{k-2}}{k(k-1)}$$

A_0, $A_1 = 0$

$A_2 = 0$ so $A_3, A_4 = 0$ & so on

$$\therefore y_2 = A_0 x$$

$$\therefore \boxed{y = c_1\left(1 + \frac{x^2}{3} + \frac{x^3}{6} + \cdots\right) + c_2 x}$$

2

c) $xy'' - y' + 4x^3 y = 0$

$$y = x^s \sum_{k=0}^{\infty} A_k x^k$$

$$\Rightarrow \quad x^s \Big[\sum_{k=0}^{\infty} (s+k)(s+k-1) A_k x^{k-1} - \sum_{k=0}^{\infty} (s+k) A_k x^{k-1} + 4 \sum_{k=0}^{\infty} A_k x^{k+3} \Big] = 0$$

$$\Rightarrow \quad x^s \Big[s(s-1) A_0 x^{-1} + (s+1) s A_1 x^0 + (s+2)(s+1) A_2 x + (s+3)(s+2) A_3 x^2 - s A_0 x^{-1} - (s+1) A_1 - (s+2) A_2 x + (s+3) A_3 x^2 + \sum_{k=4}^{\infty} \Big[(s+k)(s+k-2) A_k x^{k-1} + 4 A_{k-4} x^{k-1} \Big] \Big] = 0$$

$\therefore$ Recursive equation

$$A_k = \frac{-4 A_{k-4}}{(s+k)(s+k-2)}$$

Indicial equation

$$s(s-2) A_0 x^{-1} = 0 \quad \Rightarrow \quad s = 0, 2$$

$s = 0$

$$A_k = \frac{-4 A_{k-4}}{k(k-2)} \qquad A_1 = A_2 = A_3 = 0$$

$$A_4 = \frac{-4 A_0}{4 \cdot 2}, \quad A_5, A_6, A_7 = 0, \quad A_8 = \frac{-4 A_4}{8 \cdot 6} = \frac{4^2 A_0}{2(1 \cdot 2 \cdot 3 \cdot 4)}$$

& so on

$$y_1 = A_0\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - + \cdots\right)$$

$s=2$

$$A_k = \frac{-4A_{k-4}}{k^2+2k} \qquad A_1, A_2, A_3 = 0$$

$$A_4 = \frac{-4A_0}{4\cdot 6} = -\frac{A_0}{6}, \qquad A_5, A_6, A_7 = 0$$

$$A_8 = \frac{-4A_4}{8\cdot 10} = \frac{16A_0}{10\cdot 4\cdot 6\cdot 8} = \frac{A_0}{120} = \frac{A_0}{5!}$$

& so on

$$\therefore\ y_2 = A_0 x^2\left(1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \cdots\right)$$

$$\boxed{\begin{aligned} y = {} & c_1\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right) \\ & + c_2 x^2\left(1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \cdots\right) \end{aligned}}$$

14) $xy'' + (c-x)y' - ay = 0$; c is non-integral

$$y = \sum_{k=0}^{\infty} A_k x^{s+k} \quad ; \quad y' = \sum_{k=0}^{\infty} A_k (s+k) x^{s+k-1}$$

$$y'' = \sum_{k=0}^{\infty} A_k (s+k)(s+k-1)\, x^{s+k-2}$$

$$x^s \Big[\sum_{k=0}^{\infty} A_k x^{k-1}(s+k-1)(s+k) + c \sum_{k=0}^{\infty} A_k (s+k)\, x^{k-1} - \sum_{k=0}^{\infty} A_k (s+k)\, x^{k}$$

$$- a \sum_{k=0}^{\infty} A_k x^{k} \Big] = 0$$

<u>Indicial equation</u>

$s(s+c-1) = 0 \quad \to \quad \underline{s = 0,\ 1-c}$

Also,
$$A_k = \frac{(k+s+a-1)A_{k-1}}{(k+s)(k+s-1+c)}$$

<u>$s=0$</u>

$$A_k = \frac{(k+a-1)A_{k-1}}{k(k-1+c)}$$

A_0, $A_1 = \dfrac{(a)A_0}{c}$, $A_2 = \dfrac{(a+1)A_1}{2(c+1)} = \dfrac{a(a+1)A_0}{2c(c+1)}$

$$A_3 = \frac{(a+2)A_2}{3(2+c)} = \frac{a(a+1)(a+2)A_0}{3\cdot 2\, c(c+1)(c+2)}$$

& so on

$$\therefore\ y_1 = A_0\left(1 + \sum_{k=1}^{\infty} \frac{a(a+1)\cdots(a+k-1)}{c(c+1)\cdots(c+k-1)k!}x^k\right) = \underline{M(a,c;x)} \quad -(1)$$

$\underline{s = 1-c}$

$$A_k = \frac{(k+a-c)A_{k-1}}{(k+1-c)k}$$

$$A_0,\quad A_1 = \frac{(\cancel{k}+a-c)A_0}{(\cancel{k}+1-c)\cancel{k}}$$

$$= \frac{(1+a-c)A_0}{(2-c)}$$

$$A_2 = \frac{(2+a-c)A_1}{2(3-c)} = \frac{(1+a-c)(2+a-c)}{2(2-c)(3-c)}\cdot A_0 \quad \text{and so on}$$

$$y_2 = A_0\left(1 + \sum_{k=1}^{\infty} \frac{(1+a-c)(2+a-c)\cdots(k+a-c)}{k!\,(2-c)(3-c)\cdots(k+1-c)}x^k\right)x^{1-c} \quad -(2)$$

From (1) and (2) we see that

$$y_2 = M(1+a-c, 2-c; x)$$

$\Rightarrow$

$$\boxed{y = c_1 M(a,c;x) + c_2 x^{1-c} M(1+a-c, 2-c; x)}$$

<u>Proved</u>

Additional Problems

1) (a) $xy'' + (x-6)y' - 3y = 0$; regular singular point at $x = 0$

$$y = \sum_{k=0}^{\infty} A_k x^{s+k} \quad ; \quad y' = \sum_{k=0}^{\infty} (s+k) A_k x^{s+k-1}$$

$$y'' = \sum_{k=0}^{\infty} (s+k)(s+k-1) A_k x^{s+k-2}$$

$$x^s \Big[\sum_{k=0}^{\infty} A_k x^{k-1} (s+k)(s+k-1) + \sum_{k=0}^{\infty} (s+k) A_k \left(x^k - 6x^{k-1} \right)$$

$$- 3 \sum_{k=0}^{\infty} A_k x^k \Big] = 0$$

Incidial equation

$$s(s-1) - 6s = 0$$

$$\rightarrow \quad \underline{s = 0, 7}$$

Recursive equation

$$A_k = - \frac{(s+k-4) A_{k-1}}{(s+k)(s+k-7)}$$

$\underline{s=0}$

$$A_k = \frac{-(k-4) A_{k-1}}{k(k-7)}$$

$$A_0, \quad A_1 = \frac{-(-3)A_0)}{1 \cdot (-6)} = \frac{-A_0}{2}, \quad A_2 = \frac{-(-2)A_1}{2(-5)} = -\frac{1}{5}\left(-\frac{A_0}{2}\right) = \frac{A_0}{10}$$

$$A_3 = \frac{-(-1)A_2}{3(-4)} = \frac{-1}{12} \frac{A_0}{10} = -\frac{A_0}{120}$$

$\therefore y_1 = A_0\left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \cdots\right)$

$\underline{s=7}$

$$A_k = -\frac{(3+k)A_{k-1}}{k(k+7)}$$

$\therefore A_0, \quad A_1 = \frac{-4}{8}A_0 = \frac{-A_0}{2}$

$A_2 = -\frac{(5)}{2(9)}A_1 = \frac{5A_0}{36}, \quad A_3 = \frac{-6A_2}{3(10)} = -\frac{A_2}{5} = -\frac{A_0}{36}$

& so on

$\therefore y_2 = A_0 x^7\left(1 - \frac{x}{2} + \frac{5x^2}{36} - \frac{x^3}{36} + \cdots\right)$

$$\boxed{y = c_1\left(1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} + \cdots\right) + c_2 x^7\left(1 - \frac{x}{2} + \frac{5x^2}{36} - + \cdots\right)}$$

(b) $xy'' + y' - 4y = 0$; regular singular point at $x = 0$

$y = \sum_{k=0}^{\infty} A_k x^{s+k}$; $y' = \sum_{k=0}^{\infty} (s+k) A_k x^{s+k-1}$

$y'' = \sum_{k=0}^{\infty} (s+k)(s+k-1) A_k x^{s+k-2}$

$$\Rightarrow x^s\left[\sum_{k=0}^{\infty}(s+k)(s+k-1)A_k x^{s+k-1} + \sum_{k=0}^{\infty}(s+k)A_k x^{s+k-1} - 4\sum_{k=0}^{\infty}A_k x^{s+k}\right] = 0$$

Indicial equation

$$s(s-1) + s = 0$$

$$s^2 = 0 \Rightarrow \underline{s = 0}$$

Recursive equation $\quad A_k = \dfrac{4A_{k-1}}{k^2}$

$A_0,\ A_1 = 4A_0,\ A_2 = \dfrac{4A_1}{4} = A_1 = 4A_0,$

$A_3 = \dfrac{4A_1}{9} = \dfrac{16A_0}{9},\quad A_4 = \dfrac{4A_3}{16} = \dfrac{A_3}{4} = \dfrac{4A_0}{9}$

$$\boxed{y = c_1\left(1 + 4x + 4x^2 + \frac{16}{9}x^3 + \frac{4}{9}x^4 + \cdots\right)}$$

2) $\quad 16x^2y'' + 16xy' + (16x^2-1)y = 0$

Dividing by 16, we have,

$$x^2y'' + xy' + \left(x^2 - \frac{1}{16}\right)y = 0$$

This is a differential equation of the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

Thus, the solution is of the form,

$$y = c_1 J_p(x) + c_2 J_{-p}(x), \text{ where}$$

$J_p(x)$ = Bessel function of the first kind, of order p.

here $\underline{p = \pm \frac{1}{4}}$

$$\therefore y = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)$$

$$\therefore \boxed{y = c_1 \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+1/4}}{k!\,(k+\frac{1}{4})!} + c_2 \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-1/4}}{k!\,(k-\frac{1}{4})!}}$$

3) $xy'' + (1-2n)y' + xy = 0$ — ①

Let $y = x^n J_n(x)$ — ②

$J_n(x)$ is the solution of a D.E of the form

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

Substituting ② in ①.

$$(1-2n)\left(n x^{n-1} J_n(x) + x^n J_n'(x)\right)$$
$$+ x\left(x^n J_n''(x) + n x^{n-1} J_n'(x) + n x^{n-1} J_n'(x) + n(n-1) x^{n-2} J_n(x)\right)$$
$$+ x^{n+1} J_n(x) = 0$$

$$\Rightarrow J_n''(x)\,(x^{n+1}) + J_n'(x)\,(2nx^n + x^n - 2nx^n)$$
$$+ J_n(x)\,(n(n-1)x^{n-1} + nx^{n-1} - 2n^2x^{n-1} + x^{n+1})$$

$$= x^{n+1}J_n''(x) + x^n J_n'(x) + (x^{n+1} - n^2x^{n-1})J_n(x)$$

$$= x^{n-1}\Big[\underbrace{x^2J_n''(x) + xJ_n'(x) + (x^2-n^2)J_n(x)}_{=0\ \{\text{solution is } J_n(x)\}}\Big]$$

$$= \underline{0}$$

$\therefore$ $y = x^nJ_n(x)$ is a solution of the given differential equation ■

4) Show that Legendre's equation has alternate form

$$\cancel{\sin\theta \frac{d^2y}{d\theta^2} + \cos\theta \frac{dy}{d\theta} + n(n+1)\sin\theta\, y = 0}$$

$$\cancel{\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)\,y = 0}$$

$$\sin\theta\, y'' + \cos\theta\, y' + n(n+1)\sin\theta\, y = 0 \quad ; \quad y' = \frac{dy}{d\theta}, \quad y'' = \frac{d^2y}{d\theta^2}$$

Legendre function

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \quad \text{— (1)}$$

let $x = \cos\theta$

$dx = -\sin\theta\, d\theta$

or $\dfrac{dx}{d\theta} = -\sin\theta$

Now, $\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = -\frac{1}{\sin\theta}\frac{dy}{d\theta}$ — ②

& $\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx} = \frac{d}{d\theta}\left(-\frac{1}{\sin\theta}\frac{dy}{d\theta}\right)\left(-\frac{1}{\sin\theta}\right)$

$= \left(\frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} - \frac{1}{\sin\theta}\frac{d^2y}{d\theta^2}\right)\left(-\frac{1}{\sin\theta}\right)$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin^3\theta}\frac{dy}{d\theta}$ — ③

Substituting ② and ③ in ①,

$$\sin^2\theta\left(\frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin^3\theta}\frac{dy}{d\theta}\right) + \frac{2\cos\theta}{\sin\theta}\frac{dy}{d\theta} + n(n+1)y = 0$$

$$\Rightarrow \frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + n(n+1)y = 0$$

or $$\boxed{(\sin\theta)\frac{d^2y}{d\theta^2} + (\cos\theta)\frac{dy}{d\theta} + n(n+1)(\sin\theta)y = 0}$$

12.d) $(1-\cos x)\ y'' - \sin x\ y' + y = 0$

Let $y = \sum_{k=0}^{\infty} A_k x^{s+k}$, $y' = \sum_{k=0}^{\infty} A_k (s+k) x^{s+k-1}$, $y'' = \sum_{k=0}^{\infty} A_k (s+k)(s+k-1) x^{s+k-2}$

To find the first two terms of each series we can use

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!}$$

Then

$$\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)\left[A_0 s(s-1) x^{s-2} + A_1 (s+1) s x^{s-1} + A_2 (s+2)(s+1) x^s + \cdots\right]$$

$$-\left(x - \frac{x^3}{3!} + \cdots\right)\left[A_0 s x^{s-1} + A_1 (s+1) x^s + A_2 (s+2) x^{s+1} + \cdots\right]$$

$$+ \left[A_0 x^s + A_1 x^{s+1} + A_2 x^{s+2} + \cdots\right]$$

$$= A_0\left[\frac{s(s+1)}{2} - s + 1\right] x^s \qquad (k=0)$$

$$+ A_1\left[\frac{s(s+1)}{2} - (s+1) + 1\right] x^{s+1} \qquad (k=1)$$

$$+ \left[-A_0 \frac{s(s-1)}{4!} + A_2 \frac{(s+1)(s+2)}{2} - A_2 (s+2) + A_0 \frac{s}{3!} + A_2\right] x^{s+2} + \cdots$$

Indicial Equation:

$$\frac{s(s+1)}{2} - s + 1 = 0$$

$$\Rightarrow (s-1)(s-2) = 0$$

$$\Rightarrow s = 1, 2.$$

$\underline{s=1}$

$k=1: \; A_1\left[\frac{1(2)}{2} - (2) + 1\right] = A_1 \cdot 0 = 0$

$\Rightarrow A_1$ is arbitrary. The series with A_1 is the same for $s=2$, so I will set $A_1 = 0$.

$k=2: \; -A_0 \frac{1(0)}{4!} + A_2 \frac{(3)(2)}{2} - A_2(3) + A_0 \cdot \frac{1}{3!} + A_2 = 0$

$$\Rightarrow A_2 = -\frac{1}{6} A_0$$

So $y_1 = x\left(1 - \frac{1}{6}x^2\right)$

$\underline{s=2}$ $(k=1)$ $A_1\left[\frac{2(3)}{2} - (3) + 1\right] = 0$

$$\Rightarrow A_1 = 0$$

$k=2: \; A_0\left[-\frac{2}{4!} + \frac{8}{4!}\right] + A_2\left[6-4+1\right] = 0$

$$\Rightarrow A_2 = -\frac{1}{12} A_0$$

So $y_2 = x^2\left(1 - \frac{1}{12}x^2\right)$

$$\Rightarrow y = c_1 y_1 + c_2 y_2 = c_1\left(x - \frac{1}{6}x^3 + \cdots\right) + c_2\left(x^2 - \frac{1}{12}x^4 + \cdots\right)$$