6.14.80:

Show that equations (123), (126), and (127) imply the relations

$$\int\!\!\int_D (\varphi_1\nabla^2\varphi_2 - \varphi_2\nabla^2\varphi_1)\,dx\,dy = \oint_C \left(\varphi_1\frac{\partial\varphi_2}{\partial n} - \varphi_2\frac{\partial\varphi_1}{\partial n}\right)\,ds$$

$$\int\!\!\int_D [\varphi\nabla^2\varphi + (\nabla\varphi)^2]\,dx\,dy = \oint_C \varphi\frac{\partial\varphi}{\partial n}\,ds$$

$$\int\!\!\int_D \nabla^2\varphi\,dx\,dy = \oint_C \frac{\partial\varphi}{\partial n}\,ds$$

where $\varphi = \varphi(x, y)$ and $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, and where D is the region in the xy plane bounded by C.

(a) $\int\!\int_D (\varphi_1\nabla^2\varphi_2 - \varphi_2\nabla^2\varphi_1)\,dx\,dy = \oint_C \left(\varphi_1\frac{\partial\varphi_2}{\partial n} - \varphi_2\frac{\partial\varphi_1}{\partial n}\right)\,ds$

Let $\mathbf{v} = \varphi_1\nabla\varphi_2$, let $\mathbf{v}^* = \varphi_2\nabla\varphi_1$, and let $\mathbf{V} = \mathbf{v} - \mathbf{v}^*$. Then

$$\nabla\bullet\mathbf{v} = \varphi_1\nabla^2\varphi_2 + (\nabla\varphi_2)\bullet(\nabla\varphi_1)$$

$$\nabla\bullet\mathbf{v}^* = \varphi_2\nabla^2\varphi_1 + (\nabla\varphi_1)\bullet(\nabla\varphi_2)$$

$$\mathbf{V} \equiv \varphi_1\nabla\varphi_2 - \varphi_2\nabla\varphi_1$$

$$\nabla\bullet\mathbf{V} \quad = \quad \nabla\bullet(\mathbf{v} - \mathbf{v}^*) \quad = \quad \nabla\bullet\mathbf{v} - \nabla\bullet\mathbf{v}^* \quad = \quad \varphi_1\nabla^2\varphi_2 - \varphi_2\nabla^2\varphi_1.$$

Now consider equation (123):

$$\int\!\!\int\!\!\int_R (\varphi_1\nabla^2\varphi_2 - \varphi_2\nabla^2\varphi_1)\,d\tau = \oiint_S \mathbf{n}\bullet(\varphi_1\nabla\varphi_2 - \varphi_2\nabla\varphi_1)\,d\sigma$$

We know that the integrand on the left-hand side is equal to $\nabla\bullet\mathbf{V}$, and the integrand on the right-hand side is $\mathbf{V}$, so we can rewrite (123) as

$$\int\!\!\int\!\!\int_R \nabla\bullet\mathbf{V}\,d\tau = \oiint_S \mathbf{n}\bullet\mathbf{V}\,d\sigma.$$

This is simply the 3-dimensional divergence theorem. The two-dimensional divergence theorem is almost identical:

$$\int\!\!\int_D \nabla\bullet\mathbf{V}\,dA = \oint_C \mathbf{n}\bullet\mathbf{V}\,dS.$$

6.14.80 continued on next page...

Consider now the right-hand integrand:

$$\begin{aligned}\mathbf{n}\bullet\mathbf{V} &= \mathbf{n}\bullet(\varphi_1\nabla\varphi_2-\varphi_2\nabla\varphi_1)\\ &= \mathbf{n}\bullet\varphi_1\nabla\varphi_2-\mathbf{n}\bullet\varphi_2\nabla\varphi_1\\ &= \varphi_1\,\mathbf{n}\bullet\nabla\varphi_2-\varphi_2\,\mathbf{n}\bullet\nabla\varphi_1\\ &= \varphi_1(\mathbf{n}\bullet\nabla\varphi_2)-\varphi_2(\mathbf{n}\bullet\nabla\varphi_1)\\ &= \varphi_1\left(\frac{\partial\varphi_2}{\partial n}\right)-\varphi_2\left(\frac{\partial\varphi_1}{\partial n}\right).\end{aligned}$$

Hence we can say that

$$\int\!\!\int_D \nabla\bullet\mathbf{V}\,dA=\oint_C \mathbf{n}\bullet\mathbf{V}\,dS.$$

$$\int\!\!\int_D(\varphi_1\nabla^2\varphi_2-\varphi_2\nabla^2\varphi_1)\,dx\,dy=\oint_C\left(\varphi_1\frac{\partial\varphi_2}{\partial n}-\varphi_2\frac{\partial\varphi_1}{\partial n}\right)ds$$

□

(b) $\int\int_D[\varphi\nabla^2\varphi+(\nabla\varphi)^2]\,dx\,dy=\oint_C\varphi\frac{\partial\varphi}{\partial n}\,ds$

Equation (126) reads

$$\int\!\!\int\!\!\int_R[\varphi\nabla^2\varphi+(\nabla\varphi)^2]\,d\tau=\oiint_S\varphi\frac{\partial\varphi}{\partial n}\,d\varphi$$

The left-hand integrand is

$$\begin{aligned}\varphi\nabla^2\varphi+(\nabla\varphi)^2 &= \varphi\nabla\bullet\nabla+\nabla\varphi\bullet\nabla\varphi\\ &= \nabla\bullet\varphi(\nabla\varphi)\\ &= \nabla\bullet(\varphi\nabla\varphi).\end{aligned}$$

The right-hand integrand is

$$\begin{aligned}\varphi\frac{\partial\varphi}{\partial n} &= \varphi(n\bullet\nabla\varphi)\\ &= n\bullet(\varphi\nabla\varphi).\end{aligned}$$

Now if we let $\mathbf{V}=(\varphi\nabla\varphi)$, then equation (126) also reduces to the 3-D divergence theorem. Thus we can insert $\mathbf{V}$ into the 2-D version and expand it out, yielding

$$\int\!\!\int_D[\varphi\nabla^2\varphi+(\nabla\varphi)^2]\,dx\,dy=\oint_C\varphi\frac{\partial\varphi}{\partial n}\,ds.$$

□

6.14.80 continued on next page...

[redacted]

$\iint_D \nabla^2\varphi\,dx\,dy = \oint_C \frac{\partial\varphi}{\partial n}\,ds$

Equation (127) reads

$$\int\!\!\int\!\!\int_R \nabla^2\varphi\,d\tau = \underset{S}{\oiint} \frac{\partial\varphi}{\partial n}\,d\sigma.$$

If we examine the integrands here, we see that $\nabla^2\varphi = \nabla\bullet\nabla\varphi$ and $\frac{\partial\varphi}{\partial n} = \mathbf{n}\bullet\nabla\varphi$. Thus if we let $\mathbf{V} = \nabla\varphi$, this equation again simplifies to divergence theorem. Plug it into the 2-D version to obtain

$$\int\!\!\int_D \nabla^2\varphi\,dx\,dy = \oint_C \frac{\partial\varphi}{\partial n}\,ds.$$

□

✓

6.14.81

Want to verify that

$$\iint_D \left[\phi\nabla^2\phi + (\nabla\phi)^2\right] dxdy = \oint_c \phi\frac{\partial\phi}{\partial n}ds$$

When $\phi = x$ and D is the circular disk of radius a with centre at the origin. Note that

$$\frac{\partial\phi}{\partial n} = \frac{\partial x}{\partial n} = \frac{\partial x}{\partial r} = \cos\theta$$

Left-hand side: $\phi = x, \nabla^2\phi = \nabla^2 x = 0, (\boldsymbol{\nabla}\phi)^2 = (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi) = \mathbf{i}\cdot\mathbf{i} = 1$

This leaves

$$\iint_D dxdy = \text{area of circle of radius a} = \pi a^2$$

Right-hand side: $\phi = x, \dfrac{\partial\phi}{\partial n} = \cos\theta$ and put $x = a\cos\theta$ to give

$$\begin{aligned}\oint_c \phi\frac{\partial\phi}{\partial n}ds &= \int_0^{2\pi} a\cos^2\theta\cdot a\,d\phi \text{ since } ds = a d\theta \\ &= \frac{a^2}{2}\int_0^{2\pi}(\cos 2\theta + 1)\,d\theta \\ & \qquad\quad 0 \\ &= \frac{a^2}{2}\left[\frac{1}{2}\sin 2\theta + \theta\,\Big|_0^{2\pi}\right] \\ &= \pi a^2\end{aligned}$$

ie L.H.S. $=$ R.H.S.

We get

$$\begin{aligned}
\oint_C \mathbf{v}\cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin^2\theta + 2\cos^2\theta)\, d\theta \\
&= \int_0^{2\pi} (-\frac{1}{2} + \frac{1}{2}\cos 2\theta + 1 + \cos 2\theta)\, d\theta \\
&= \int_0^{2\pi} \left(+\frac{1}{2} + \frac{3}{2}\cos 2\theta\right)\, d\theta \\
&= \frac{1}{2}\theta + \frac{3}{4}\sin 2\theta \bigg|_0^{2\pi} \\
&\Rightarrow \oint_C \mathbf{v}\cdot d\mathbf{r} = \pi
\end{aligned}$$

i.e. L.M.S. = R.H.S.

29)

$$\iint_S \hat{\mathbf{n}}\cdot(\nabla\times\mathbf{v})\, d\sigma = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

$$\mathbf{v} = y\mathbf{i}$$

$$\begin{aligned}
\oint_C \mathbf{v}\cdot d\mathbf{r} &= \oint_C y dx \qquad \text{on unit circle } x^2+y^2=1 \\
&= \int_0^{2\pi} \sin^2\theta d\theta = -\frac{1}{2}\int_0^{2\pi} d\theta = -\pi
\end{aligned}$$

from (88)

6.16.88:

Verify the truth of Stokes' theorem, as given in Equation (133), in the case when $\mathbf{V} = y\,\mathbf{i} + 2x\,\mathbf{j} + z\,\mathbf{k}$, if C is the circle $x^2 + y^2 = 1$ (or $x = \cos t,\ y = \sin t$) in the xy plane, and S is the plane area bounded by C.

Equation (133) reads

$$\iint_S \mathbf{n} \bullet (\nabla \times \mathbf{V})\, d\sigma = \oint_C \mathbf{V} \bullet d\mathbf{r}.$$

First we will integrate the left-hand side. Consider the integrand

$$\mathbf{n} \bullet (\nabla \times \mathbf{V}).$$

First note that

$$\begin{aligned} \nabla \times \mathbf{V} &= \mathbf{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \\ &= \mathbf{i}\,(0-0) + \mathbf{j}\,(0-0) + \mathbf{k}\,(2-1) \\ &= 0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} \\ &= \mathbf{k}. \end{aligned}$$

Now $\mathbf{n}$ is the unit vector that is normal to the surface. The surface in question is a subset of the xy plane, so we need do no calculations; the unit normal is $\mathbf{n} = \mathbf{k}$.

Our expectation is as follows: since $\mathbf{n}$ and $\nabla \times \mathbf{V}$ are both $\mathbf{k}$, the integrand will simplify to 1, leaving the integral on the left-hand side integrating the area of the unit circle: π.

$$\begin{aligned} \iint_S 1\, d\sigma &= \int_0^{2\pi}\int_0^1 r\, dr\, d\theta \\ &= \int_0^{2\pi}\left(\left.\frac{r^2}{2}\right|_0^1\right) d\theta \\ &= \int_0^{2\pi}\frac{1}{2}\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} d\theta \\ &= \frac{1}{2}2\pi \\ &= \pi. \end{aligned}$$

6.16.88 continued on next page...

Now let us consider the integral on the right-hand side:

$$\oint \mathbf{V} \bullet d\mathbf{r}$$

Since we will be parameterizing the circle, it is helpful to recall that

$$x = \cos t \Rightarrow dx = -\sin t\, dt \qquad y = \sin t \Rightarrow dy = \cos t\, dt.$$

Using equation (82), we find that

$$\begin{aligned}\mathbf{V} \bullet d\mathbf{r} &= V_1\, dx + V_2\, dy + V_3\, dz \\ &= y\, dx + 2x\, dy + z\, dz \\ &= \sin t(-\sin t) + 2\cos t(\cos t) + 0 \\ &= 2\cos^2 t - \sin^2 t.\end{aligned}$$

Since in this case $\sin^2 t + \cos^2 t = 1$, we can rewrite our integrand as

$$2\cos^2 t - (1 - \cos^2 t)\, dt = 3\cos^2 t - 1\, dt.$$

Now we integrate this over the unit circle:

$$\begin{aligned}\oint_C \mathbf{V} \bullet d\mathbf{r} &= \int_0^{2\pi} 3\cos^2 t - 1\, dt \\ &= \int_0^{2\pi} 3\cos^2 t\, dt - \int_0^{2\pi} 1\, dt \\ &= 3\int_0^{2\pi} \cos^2 t\, dt - 2\pi. \qquad (1)\end{aligned}$$

Now to calculate the integral of $\cos^2 t\, dt$, we will need to integrate by parts:

$$\begin{aligned}\int_0^{2\pi} \cos^2 t\, dt &= \cos t \sin t\Big|_0^{2\pi} + \int_0^{2\pi} \sin^2 t\, dt \\ &= 0 + \int_0^{2\pi} (1 - \cos^2 t)\, dt \\ &= \int_0^{2\pi} 1\, dt - \int_0^{2\pi} \cos^2 t\, dt \\ &= 2\pi - \int_0^{2\pi} \cos^2 t\, dt \\ 2\int_0^{2\pi} \cos^2 t\, dt &= 2\pi \\ \int_0^{2\pi} \cos^2 t\, dt &= \pi.\end{aligned}$$

6.16.88 continued on next page...

Thus our integral (1) can be solved:

$$\begin{aligned}\oint_C \mathbf{V} \bullet d\mathbf{r} &= 3\int_0^{2\pi} \cos^2 t\, dt - 2\pi \\ &= 3(\pi) - 2\pi \\ &= \pi.\end{aligned}$$

Now we see that for the given conditions, Stokes' theorem holds:

$$\int\int_S \mathbf{n} \bullet (\nabla \times \mathbf{V})\, d\sigma = \pi = \oint_C \mathbf{V} \bullet d\mathbf{r}.$$

□

6.16.93

Stoke's Theorem is given by

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\sigma$$

Want to apply this to the following:

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\alpha \frac{\partial}{\partial t} \iint_S \mathbf{H} \cdot d\sigma, \quad \oint_C \mathbf{H} \cdot d\mathbf{r} = \beta \frac{\partial}{\partial t} \iint_S \mathbf{E} \cdot d\sigma$$

Thus,

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot d\sigma$$

and

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{H}) \cdot d\sigma$$

Thus,

$$\iint_S \left(\nabla \times \mathbf{E} + \alpha \frac{\partial \mathbf{H}}{\partial t}\right) \cdot d\sigma = 0, \quad \iint_S \left(\nabla \times \mathbf{H} - \beta \frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\sigma = 0$$

But the region S is totally arbitrary so if the integral is to be zero in general, the integrand must be identically zero. Hence,

$$\nabla \times \mathbf{E} = -\alpha \frac{\partial \mathbf{H}}{\partial t} \text{and } \nabla \times \mathbf{H} = \beta \frac{\partial \mathbf{E}}{\partial t}$$

6.16.97:

For the coordinates of Problem 96 derive the relations analogous to those of Equations (168b-e) for circular cylindrical coordinates. In particular, verify that

$$\begin{aligned} h_u = h_v &= a\sqrt{\cosh^2 u - \cos^2 v}, \quad h_z = 1, \\ \mathbf{u}_1 &= \frac{\mathbf{i}\sinh u\cos v + \mathbf{j}\cosh u\sin v}{\sqrt{\cosh^2 u - \cosh^2 v}}, \\ \mathbf{u}_2 &= \frac{-\mathbf{i}\cosh u\sin v + \mathbf{j}\sinh u\cos v}{\sqrt{\cosh^2 u - \cosh^2 v}}, \\ \nabla f &= \frac{1}{a\sqrt{\cosh^2 u - \cos^2 v}}\left(\mathbf{u}_1\frac{\partial f}{\partial u} + \mathbf{u}_2\frac{\partial f}{\partial v}\right) + \mathbf{k}\frac{\partial f}{\partial z}, \\ \nabla^2 f &= \frac{1}{a^2(\cosh^2 u - \cos^2 v)}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) + \frac{\partial^2 f}{\partial z^2}. \end{aligned}$$

Show also that *for large values of u* there follows

$$\mathbf{u}_1 \sim \mathbf{i}\cos v + \mathbf{j}\sin v, \qquad \mathbf{u}_2 \sim -\mathbf{i}\sin v + \mathbf{j}\cos v,$$

and

$$\mathbf{u}_1\cos v - \mathbf{u}_2\sin v \sim \mathbf{i}, \qquad \mathbf{u}_1\sin v + \mathbf{u}_2\cos v \sim \mathbf{j}.$$

Problem 96 defines *elliptical cylindrical coordinates* by the equations

$$x = a\cosh u\cos v, \qquad y = a\sinh u\sin v, \qquad z = z, \qquad \text{for } u \geqq 0 \text{ and } 0 \leqq v < 2\pi.$$

Our coordinates will look like (u, v, z), and the position vector is

$$\mathbf{r} = a\cosh u\cos v\ \mathbf{i} + a\sinh u\sin v\ \mathbf{j} + z\ \mathbf{k}.$$

We will start with finding the basic vectors $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$. First we need the directions:

$$\begin{aligned} \mathbf{U}_1 &= \frac{\partial \mathbf{r}}{\partial u} = (a\cos v\sinh u)\,\mathbf{i} + (a\sin v\cosh u)\,\mathbf{j}. \\ \mathbf{U}_2 &= \frac{\partial \mathbf{r}}{\partial v} = (-a\cosh u\sin v)\,\mathbf{i} + (a\sinh u\cos v)\,\mathbf{j}. \\ \mathbf{U}_3 &= \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}. \end{aligned}$$

6.16.97 continued on next page...

Now since $\mathbf{U}_1 = h_1\,\mathbf{u}_1$, we need to normalize $\mathbf{U}_1$ by its length, that is, find h_1:

$$\begin{aligned}
h_1 &= |\,\mathbf{u}_1| \\
&= \sqrt{(a\cos v\sinh u)^2 + (a\sin v\cosh u)^2} \\
&= \sqrt{a^2\cos^2 v\sinh^2 u + a^2\sin^2 v\cosh^2 u} \\
&= a\sqrt{\cos^2 v\sinh^2 u + \sin^2 v\cosh^2 u} \\
&= a\sqrt{\cos^2 v\sinh^2 u + (1-\cos^2 v)\cosh^2 u} \\
&= a\sqrt{\cos^2 v\sinh^2 u + \cosh^2 u - \cos^2 v\cosh^2 u} \\
&= a\sqrt{\cosh^2 u - \cos^2 v\cosh^2 u + \cos^2 v\sinh^2 u} \\
&= a\sqrt{\cosh^2 u - \cos^2 v(\cosh^2 u - \sinh^2 u)} \\
&= a\sqrt{\cosh^2 u - \cos^2 v(1)} \\
&= a\sqrt{\cosh^2 u - \cos^2 v}\,. \quad \checkmark
\end{aligned}$$

Similarly, we will need to find h_2:

$$\begin{aligned}
h_2 &= |\,\mathbf{u}_2| \\
&= \sqrt{(-a\cosh u\sin v)^2 + (a\sinh u\cos v)^2} \\
&= \sqrt{a^2\cosh^2 u\sin^2 v + a^2\sinh^2 u\cos^2 v} \\
&= a\sqrt{\cosh^2 u\sin^2 v + \sinh^2 u\cos^2 v} \\
&= a\sqrt{\cosh^2 u - \cos^2 v}\,. \\
&= h_1. \quad \checkmark
\end{aligned}$$

The norm of $\mathbf{U}_3$ is easy, it's already a unit vector, $\mathbf{k}$, so $h_3 = 1$. ✓
Since $\mathbf{u}_1 = \dfrac{\mathbf{U}_1}{h_1}$ and $\mathbf{u}_2 = \dfrac{\mathbf{U}_2}{h_2}$, we can see that

$$\mathbf{u}_1 = \frac{\cos v\sinh u\;\mathbf{i} + \sin v\cosh u\;\mathbf{j}}{\sqrt{\cosh^2 u - \cos^2 v}}. \quad \checkmark$$

$$\mathbf{u}_2 = \frac{-\cosh u\sin v\;\mathbf{i} + \sinh u\cos v\;\mathbf{j}}{\sqrt{\cosh^2 u - \cos^2 v}}. \quad \checkmark$$

6.16.97 continued on next page...

We can use equation (157) to find ∇f:

$$\begin{aligned}
\nabla f &= \frac{\mathbf{u}_1}{h_1}\frac{\partial f}{\partial u_1} + \frac{\mathbf{u}_2}{h_2}\frac{\partial f}{\partial u_2} + \frac{\mathbf{u}_3}{h_3}\frac{\partial f}{\partial u_3} \\
&= \frac{\mathbf{u}_1}{h_1}\frac{\partial f}{\partial u_1} + \frac{\mathbf{u}_2}{h_1}\frac{\partial f}{\partial u_2} + \mathbf{u}_3\frac{\partial f}{\partial u_3} \\
&= \frac{1}{h_1}\left(\mathbf{u}_1\frac{\partial f}{\partial u_1} + \mathbf{u}_2\frac{\partial f}{\partial u_2}\right) + \mathbf{u}_3\frac{\partial f}{\partial u_3} \\
&= \frac{1}{a\sqrt{\cosh^2 u - \cos^2 v}}\left(\mathbf{u}_1\frac{\partial f}{\partial u} + \mathbf{u}_2\frac{\partial f}{\partial v}\right) + \mathbf{u}_3\frac{\partial f}{\partial z}. \quad ✓
\end{aligned}$$

It follows from equation (166) that $\nabla^2 f$ is

$$\begin{aligned}
\nabla^2 f &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right] f \\
&= \frac{1}{h_1^2}\left[\frac{\partial}{\partial u_1}\left(\frac{h_1}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1}{h_1}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(h_1^2\frac{\partial}{\partial u_3}\right)\right] f \\
&= \left\{\frac{1}{h_1^2}\left[\frac{\partial}{\partial u_1}\left(\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{\partial}{\partial u_2}\right)\right] + \frac{\partial}{\partial u_3}\left(\frac{\partial}{\partial u_3}\right)\right\} f \\
&= \left\{\frac{1}{h_1^2}\left[\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right] + \frac{\partial^2}{\partial u_3^2}\right\} f \\
&= \frac{1}{a^2(\cosh^2 u - \cos^2 v)}\left[\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right] + \frac{\partial^2 f}{\partial z^2}. \quad ✓
\end{aligned}$$

Now about those hyperbolic trig functions... They are defined as

$$\sinh(u) = \frac{e^u - e^{-u}}{2} \qquad \text{and} \qquad \cosh(u) = \frac{e^u + e^{-u}}{2}.$$

The intuitive argument is simplest. As u gets "huge", sinh and cosh approach each other asymptotically:

$$\sinh(huge) \to \frac{e^{huge} - e^{-huge}}{2} \to \frac{huge - tiny}{2} \to \frac{huge}{2}$$

$$\cosh(huge) \to \frac{e^{huge} + e^{-huge}}{2} \to \frac{huge + tiny}{2} \to \frac{huge}{2}$$

Since cos is bounded between 0 and 1, the cos term in the denominator will be swamped by the unbounded cosh term. Thus the hyperbolic terms in the numerators will be "cancelled" by the cosh term in the denominator, leaving

$$\mathbf{u}_1 \sim \mathbf{i}\cos v + \mathbf{j}\sin v, \qquad \mathbf{u}_2 \sim -\mathbf{i}\sin v + \mathbf{j}\cos v. \quad ✓$$

6.16.97 continued on next page...

$$\begin{aligned}
\mathbf{u}_1 \cos v &\sim \cos^2 v \ \mathbf{i} + \sin v \cos v \ \mathbf{j} \\
-\mathbf{u}_2 \sin v &\sim \sin^2 v \ \mathbf{i} - \sin v \cos v \ \mathbf{j} \\
\mathbf{u}_1 \cos v - \mathbf{u}_2 \sin v &\sim \cos^2 v \ \mathbf{i} + \sin v \cos v \ \mathbf{j} + \sin^2 v \ \mathbf{i} - \sin v \cos v \ \mathbf{j} \\
&\sim (\sin^2 v + \cos^2 v) \ \mathbf{i} \\
&\sim \mathbf{i}. \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
\mathbf{u}_1 \sin v &\sim \sin v \cos v \ \mathbf{i} + \sin^2 v \ \mathbf{j} \\
\mathbf{u}_2 \cos v &\sim -\sin v \cos v \ \mathbf{i} + \cos^2 v \ \mathbf{j} \\
\mathbf{u}_1 \sin v + \mathbf{u}_2 \cos v &\sim \sin v \cos v \ \mathbf{i} + \sin^2 v \ \mathbf{j} - \sin v \cos v \ \mathbf{i} + \cos^2 v \ \mathbf{j} \\
&\sim (\sin^2 v + \cos^2 v) \ \mathbf{j} \\
&\sim \mathbf{j}. \quad \checkmark
\end{aligned}$$

$\square$