HW3
Solutions

60)

a) $\mathbf{F} = 2xyz^3\,\hat{\imath} - (x^2z^3 + 2y)\,\hat{\jmath} + 3x^2yz^2\hat{\mathbf{k}}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{\imath}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \hat{\imath}\,(3x^2z^2 + 3x^2z^2) - \hat{\jmath}\,\overset{0}{(6xyz^2 - 6xyz^2)} + \hat{\mathbf{k}}\,(-2xz^3 - 2xz^3)$$

$\neq 0$ – no solution

b) $\mathbf{F} = 2xy\hat{\imath} + (x^2 + 2yz)\,\hat{\jmath} + (y^2 + 1)\,\hat{\mathbf{k}}$

$$\nabla \times \mathbf{F} = \hat{\imath}\overset{0}{(2y - 2y)} - \hat{\jmath}\,(0 - 0) + \hat{\mathbf{k}}\,\overset{0}{(2x - 2x)}$$

$$= 0$$

$$\mathbf{F} \cdot d\mathbf{r} = 2xy\,dx + (x^2 + 2yz)\,dy + (y^2 + 1)\,dz$$

$$= \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz$$

$$\frac{\partial \phi}{\partial x} = 2xy \quad \phi(x,y,z) = x^2y + f\,(y,z)$$

$\swarrow$

$$\frac{\partial \phi}{\partial y} = (x^2 + 2yz) = x^2 + \frac{\partial f}{\partial y} \Rightarrow \frac{\partial f}{\partial y} = 2yz, \quad f\,(y,z) = y^2z + g(z)$$

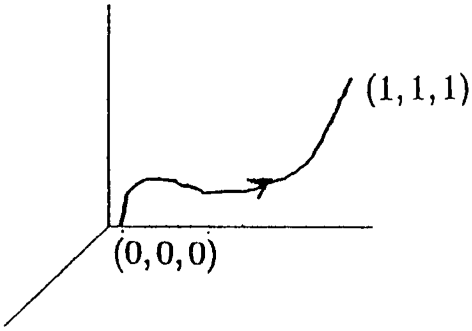
$$\phi(x,y,z) = x^2y + y^2z + g(z)$$

$$\frac{\partial \phi}{\partial z} = y^2 + 1 = y^2 + \frac{\partial g}{\partial z} \Rightarrow \frac{\partial g}{\partial z} = 1,\ g = z + \text{constant}$$

$$\phi(x,y,z) = x^2y + y^2z + z + c$$

61)

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \int_c (F_x \cdot dx + F_y dy + F_z dz)$$



$$y = x^2,\ \ z = x^3$$

$$dy = 2xdx$$

$$dz = 3x^2dx$$

(a)

$$\vec{F} = 2xyz^3\hat{\imath} - (x^2z^3 + 2y)\,\hat{\jmath} + 3x^2yz^2\hat{\mathbf{k}}$$

$$\begin{aligned}
&\int_0^1 \left\{2xx^2x^9 - (x^2x^9 + 2x^2)2x + (3x^2x^2x^6) - 3x^2\right\} dx \\
=\ &\int_0^1 (2x^{12} - 2x^{12} - 4x^3 + 9x^2)dx \\
=\ &-4\int_0^1 x^3dx + 9\int_0^1 x^{12}dx \\
=\ &-4\cdot\frac{1}{4} = \frac{9}{13} = -1 + 9/13 = -4/13
\end{aligned}$$

(b)

$$\mathbf{F} = 2xy\,\hat{\imath} + (x^2 + 2yz)\,\hat{\jmath} + (y^2 + 1)\,\hat{\mathbf{k}}$$

$$\int_0^1 \{2xx^2 + (x^2 + 2x^2x^3)2x + (x^4 + 1)\,3x^2\}\,dx$$

$$= \int_0^1 (2x^3 + 2x^3 + 4x^6 + 3x^6 + 3x^2)\,dx$$

$$= \int_0^1 (4x^3 + 7x^6 + 3x^2)\,dx = 1 + 1 + 1 = 3$$

67. $\underline{F} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$

$\varphi = z - xy - 1$

$\rightarrow \quad \nabla\varphi = -y\hat{\underline{i}} - x\hat{\underline{j}} + \hat{\underline{k}} \qquad |\nabla\varphi| = \sqrt{y^2 + x^2 + 1}$

$\hat{\underline{n}} = \pm \dfrac{-y\hat{\underline{i}} - x\hat{\underline{j}} + \hat{\underline{k}}}{\sqrt{x^2+y^2+1}} \qquad \cos\gamma = \hat{\underline{n}}\cdot\hat{\underline{k}} = \dfrac{1}{\sqrt{x^2+y^2+1}}$

$$\iint_S \hat{\underline{F}}\cdot\hat{\underline{n}}\, d\sigma = \int_0^1\!\!\int_0^1 \frac{(-xy \quad -xy + z)}{\sqrt{x^2+y^2+1}}\, dx\,dy \,\sqrt{x^2+y^2+1}$$

$$= \int_0^1\!\!\int_0^1 (-2xy + (xy+1))\, dx\,dy$$

$$= \int_0^1\!\!\int_0^1 (1 - xy)\, dx\,dy$$

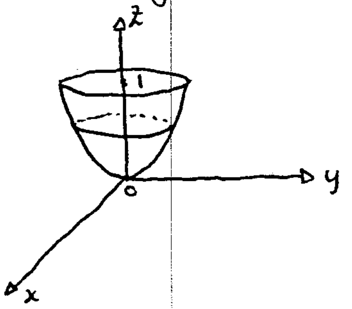
$$= \int_0^1\!\!\int_0^1 dx\,dy - \int_0^1\!\!\int_0^1 xy\, dx\,dy$$

$$= 1 - \int_0^1 x\,dx \int_0^1 y\,dy$$

$$= 1 - \tfrac{1}{2}\cdot\tfrac{1}{2}$$

$$= \frac{3}{4}.$$

6.12.70 Evaluate surface integral of $\underline{F} = (yz, xz, xy)$ over the closed boundary of the region bounded below by $z = x^2 + y^2$ and above by $z = 1$.



$$\varphi = z - x^2 + y^2$$

$$\underline{\hat{n}} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2x\underline{\hat{i}} - 2y\underline{\hat{j}} + \underline{\hat{k}}}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\cos\gamma = \underline{\hat{n}} \cdot \underline{\hat{k}} = \frac{-1}{\sqrt{4x^2+4y^2+1}}$$

$$\iint \underline{F} \cdot \underline{n}\, d\sigma = \iint (-2xyz - 2xyz + xy)\, dx\,dy$$

$$= \iint (-4xyz + xy)\, dx\,dy$$

$$= \iint (-4xy(x^2+y^2) + xy)\, dx\,dy.$$

Polar co-ordinates: $x = r\cos\theta$, $y = r\sin\theta$ $\qquad x^2 + y^2 = r^2$, $xy = r^2\cos\theta\sin\theta$

$$\Rightarrow \iint \underline{F} \cdot \underline{n}\, d\sigma = \int_0^{2\pi}\int_0^1 r^2\cos\theta\sin\theta(-4r^2+1)\,r\,dr\,d\theta$$

This solution only finds flux over bottom surface, not capping disc

$$= \int_0^{2\pi} \cos\theta\sin\theta\, d\theta \int_0^1 (-4r^5 + r^3)\,dr.$$

$$= \overset{0}{\cancel{\frac{\cos 2\theta}{4}\Big|_0^{2\pi}}} \to 0 \cdot \int_0^1 (-4r^5 + r^3)\,dr$$

$$= 0$$

72. $\underline{F} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$

$\rightarrow \quad \nabla\cdot\underline{F} = 1 + 1 + 1 = 3.$

$$\iint_S \underline{F}\cdot\underline{n}\,d\sigma = \iiint_V \nabla\cdot\underline{F}\,d\tau$$ (Divergence Theorem).

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 3\,dx\,dy\,dz = 3\int_{-1}^{1}dx\int_{-1}^{1}dy\int_{-1}^{1}dz$$

$$= 3.2.2.2$$

$$= 24$$

6.13.73

Determine the value of the surface integral $\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma$ in each of the following cases, by use of the divergence theorem.

(a) $\mathbf{F}=(x,y,z)$; S is the closed spherial surface $x^2+y^2+z^2=1$.

By divergence theorem

$$\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma=\iiint\limits_R \nabla\cdot\mathbf{F}\,d\tau$$

with

$$\iiint\limits_R \boldsymbol{\nabla}\cdot\mathbf{F}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(x,y,z)=1+1+1=3 \quad \text{we get}$$

$$\iiint\limits_R \boldsymbol{\nabla}\cdot\mathbf{F}d\tau=3\iiint\limits_{\text{sphere}} d\tau=3\cdot\frac{4\pi}{3}=4\pi$$

(b) $\mathbf{F}=(xy,xz,1-z-yz)$; S is the closed surface composed of the portion of the paraboloid $z=1-x^2-y^2$ for which $z\geq 0$ and the circular disk $x^2+y^2=1$, $z=0$. By divergence theorem,

$$\iiint\limits_R \boldsymbol{\nabla}\cdot\mathbf{F}d\tau=\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma$$

with

$$\boldsymbol{\nabla}\cdot\mathbf{F}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(xy,xz,1-z-yz)=y+0-1-y=-1$$

Thus,

$$\iiint\limits_R \boldsymbol{\nabla}\cdot\mathbf{F}d\tau = -\iiint\limits_{\text{parabolid}} d\tau = -\frac{\pi r^2 h}{2} = \frac{-\pi}{2}$$

(c) $\mathbf{F} = (xy, xz, 1 - z - yz)$; S is the portion of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$. By divergence theorem

$$\iint\limits_S \mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma = \iiint\limits_R \boldsymbol{\nabla}\cdot\mathbf{F}d\tau$$

But we only want one part of the surface integral.

We want the integral over S_1, given by

$$\iint\limits_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma = \iiint\limits_R \boldsymbol{\nabla}\cdot\mathbf{F}d\tau - \iint\limits_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma$$

On $S_2\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ and $\mathbf{F}\cdot\hat{\mathbf{n}} = -(1 - z - yz)$. Thus,

$$\iint\limits_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma = \iint\limits_{S_2} -(1 - z - yz)\Big|_{z=0} d\sigma = -\iint\limits_{S_2} d\sigma = -\pi$$

Since $S_2$is the unit circle. From part (b) we have,

$$\iiint_R \nabla \cdot \mathbf{F} d\tau = -\frac{\pi}{2}$$

This gives

$$\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}} \, d\sigma = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$$

(d) $\mathbf{F} = (x^2, -(1+2x), z)$; S is the lateral surface of that portion of the cylinder $x^2 + y^2 = 1$ for which $0 \geq z \leq 1$

By divergence theorem

$$\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}} \, d\sigma + \iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}} \, d\sigma + \iint_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}} \, d\sigma = \iiint_R \nabla \cdot \mathbf{F} d\tau$$

and the integral over S_2 is the one we want.

Integral over R :

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot x^2, -(1+2x), \, z) = 2x - 0 + 1 = 1 + 2x$$

Put $x = r\cos\theta$ and the volume element $d\tau = r \, d\theta \, dr \, dz$

Thus,

$$
\begin{aligned}
\iiint\limits_R (1+2x)\,d\tau &= \int\limits_{z=0}^{1}\int\limits_{\theta=0}^{2\pi}\int\limits_{r=0}^{1}(1+2r\cos\theta)r\,d\theta\,dr\,dz \\
&= \int\limits_{0}^{2\pi}\int\limits_{0}^{1}(1+2r\cos\theta)r\,dr\,d\theta \\
&= \int\limits_{0}^{2\pi}\left[\frac{r^2}{2}+\frac{2r^3}{3}\cos\theta\,\Big|_0^1\right]d\theta \\
&= \int\limits_{0}^{2\pi}(\frac{1}{2}+\frac{2}{3}\cos\theta)d\theta \\
&= \frac{1}{2}\theta+\frac{2}{3}\sin\theta\,\Big|_0^{2\pi} = \pi
\end{aligned}
$$

On S_1, $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ and $z = 1$.

$$\mathbf{F}\cdot\hat{\mathbf{n}} = (x^2, -(1+2x),\, z)\cdot(0,0,\hat{\mathbf{k}}) = z = 1.$$

The integral becomes,

$$\iint\limits_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma = \iint\limits_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,dx\,dy = \iint\limits_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,dx\,dy = \pi \text{ since circle is unit.}$$

On $S_3, \hat{\mathbf{n}} = -\hat{\mathbf{k}}$ and $z = 0$.

$$\begin{aligned}\mathbf{F} \cdot \hat{\mathbf{n}} &= (x^2, -(1+2x),\, z) \cdot (0, 0, \hat{\mathbf{k}}) = -\, z = 0. \\ &\Rightarrow \iint_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, d\sigma = 0\end{aligned}$$

So the integral becomes,

$$\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, d\sigma = \pi - \pi = 0$$

74 a) $\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$

$$\oint\!\!\oint_S \underline{r}\,.\,\underline{n}\,ds = \oint\!\!\oint\!\!\oint_V \underline{\nabla}\cdot\underline{r}\,d\tau$$

$$\nabla\cdot\underline{r} = 1+1+1 = 3$$

$$\Rightarrow \oint\!\!\oint\!\!\oint_V \nabla\cdot\underline{r}\,d\tau = \oint\!\!\oint\!\!\oint_V 3\,d\tau = 3\oint\!\!\oint\!\!\oint_V d\tau = \underline{3V.}$$

b). $$\oint\!\!\oint_S x\underline{r}\cdot\underline{n}\,d\sigma = \oint\!\!\oint\!\!\oint_V \nabla\cdot(x\underline{r})\,d\tau$$

Put $\underline{F} = x\underline{r} = x^2\hat{\underline{i}} + xy\hat{\underline{j}} + xz\hat{\underline{k}}$

$$\nabla\cdot\underline{F} = 2x + x + x$$
$$= 4x.$$

$$\Rightarrow \oint\!\!\oint_S x\underline{r}\cdot\underline{n}\,d\sigma = \oint\!\!\oint\!\!\oint_V \nabla\cdot\underline{F}\,d\tau = \oint\!\!\oint\!\!\oint_V 4x\,d\tau$$
$$= 4\oint\!\!\oint\!\!\oint_V x\,d\tau.$$
$$= \underline{4\bar{x}V.}$$