6.5.28 If **F** is a function of t, find the derivative of

$$\mathbf{F} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^2\mathbf{F}}{dt^2}$$

Solution

By definition,

$$\frac{d}{dt}\left[\mathbf{F} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^2\mathbf{F}}{dt^2}\right] = \frac{d\mathbf{F}}{dt} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^2\mathbf{F}}{dt^2} + \mathbf{F} \cdot \frac{d^2\mathbf{F}}{dt^2} \times \frac{d^2\mathbf{F}}{dt^2} + \mathbf{F} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^3\mathbf{F}}{dt^3}$$

By the identity $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot c$ we see that

$$\frac{d\mathbf{F}}{dt} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^2\mathbf{F}}{dt^2} = \frac{d\mathbf{F}}{dt} \times \frac{d\mathbf{F}}{dt} \cdot \frac{d^2\mathbf{F}}{dt^2} = 0$$

Also $\frac{d^2\mathbf{F}}{dt^2} \times \frac{d^2\mathbf{F}}{dt^2} = 0$ which leaves us with

$$\frac{d}{dt}\left[\mathbf{F} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^2\mathbf{F}}{dt^2}\right] = \mathbf{F} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^3\mathbf{F}}{dt^3}$$

OK

6.5.32

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ represent the position vector from a fixed origin O to a point P, and suppose that the xyz axis system is rotating about a fixed vector $\vec{\omega}$ through O, with angular velocity of constant magnitude ω.

(a) By calculating $\frac{d\mathbf{r}}{dt}$, and noticing that $\frac{d\mathbf{i}}{dt} = \omega \times i$, and so forth, obtain the velocity vector in the form

$$\mathbf{v} = \mathbf{v}_o + \omega \times \mathbf{r}$$

where the vector

$$\mathbf{v}_o = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

is the velocity vector which would be obtained if the aces were fixed.

Solution

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{\mathbf{k}} + x\frac{d\hat{\imath}}{dt} + \mathbf{y}\frac{d\hat{\jmath}}{dt} + \mathbf{z}\frac{d\hat{\mathbf{k}}}{dt}$$

because $\hat{\imath}$, $\hat{\jmath}$ and $\hat{\mathbf{k}}$ are changing in direction.

use

$$\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}$$

Therefore

$$\frac{d\vec{r}}{dt} = \vec{\mathbf{v}} = \mathbf{v}_o + \vec{\boldsymbol{\omega}} \times \vec{r}$$

$$\vec{\mathbf{v}}_o = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{\mathbf{k}}$$

is ~~the line~~ ~~or~~ velocity as in non-rotating coordinate system.

(b) Obtain the acceleration vector in the form

$$\mathbf{a} = \mathbf{a}_o + 2\omega \times \mathbf{v}_o + \omega \times (\omega \times \mathbf{r}),$$

where $\mathbf{v}_o$ is defined in part (a), and where

$$\mathbf{a}_o = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

Solution

$$\mathbf{a} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}\vec{\mathbf{v}}_o + \vec{\omega} \times \frac{d\vec{\mathbf{r}}}{dt}$$

$$= \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{\mathbf{k}} + \frac{dx}{dt} \cdot \frac{d\hat{\imath}}{dt} + \frac{dy}{dt} \cdot \frac{d\hat{\jmath}}{dt} + \frac{dz}{dt} \cdot \frac{d\hat{\mathbf{k}}}{dt}$$

$$+ \vec{\omega} \times (\vec{\mathbf{v}}_o + \vec{\omega} \times \vec{\mathbf{r}})$$

$$= \vec{\mathbf{a}} + 2\,\vec{\omega} \times \vec{\mathbf{v}} + \vec{\omega} \times \vec{\omega} \times \vec{\mathbf{r}}$$

6.6.34

(a) If θ is polar angle, show that the vectors

$$\mathbf{u}_1 = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta, \quad \mathbf{u}_2 = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$

are perpendicular unit vectors in the radial and circumferential directions, respectively, in the xy plane, and that

$$\frac{d\mathbf{u}_1}{d\theta} = \mathbf{u}_2, \quad \frac{d\mathbf{u}_2}{d\theta} = -\mathbf{u}_1.$$

solution

(a) $\mathbf{u}_1 = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$\mathbf{u}_2 = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

$$\|\mathbf{u}_1\| = (\cos^2\theta + \sin^2\theta)^{\frac{1}{2}} = 1$$
$$\|\mathbf{u}_2\| = (\sin^2\theta + \cos^2\theta)^{\frac{1}{2}} = 1$$
$\Rightarrow$ $\mathbf{u}_1, \mathbf{u}_2$ are unit vectors

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

$\Rightarrow$ $\mathbf{u}_1, \mathbf{u}_2$ are perpendicular (in the radial and circumferential directions respectively).

$$\frac{d\mathbf{u}_1}{d\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = \mathbf{u}_2$$

bold

$$\frac{d\mathbf{u}_2}{d\theta} = -\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} = \mathbf{u}_1 \quad -(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -\mathbf{u}_1$$

(b) For points on a plane curve in polar coordinates, the position vector is of the form $\mathbf{r} = r\mathbf{u}_1$. By differentiation with respect to time t, obtain expressions for the vectors of velocity and acceleration of a point moving along the curve and show that the radial and circumferential components are of the form

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta},$$
$$a_r = \ddot{r} - r\dot{\theta}^2, \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}),$$

where a dot denotes time differentiation. [Notice, for example, that $\dot{\mathbf{u}}_1 = (\frac{d\mathbf{u}_1}{d\theta})\dot{\theta}$.]

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{u}_1 + r\frac{d\mathbf{u}_1}{d\theta}\dot{\theta} \qquad \mathbf{r} = r\mathbf{u}_1$$

$$= \dot{r}\mathbf{u}_1 + r\dot{\theta}\mathbf{u}_2 \qquad \dot{\mathbf{u}}_1 = \frac{d\mathbf{u}_1}{d\theta}\dot{\theta}$$

$$\mathbf{v} = \quad = V_r\vec{u}_1 + V_\theta\vec{u}_2 = V_r\hat{r} + V_\theta\hat{\theta}$$

$$\Rightarrow v_r = \dot{r} \qquad v_\theta = r\dot{\theta}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\dot{r}\mathbf{u}_1 + r\dot{\theta}\mathbf{u}_2)$$

$$= \ddot{r}\mathbf{u}_1 + \dot{r}\frac{d\mathbf{u}_1}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\mathbf{u}_2 + r\ddot{\theta}\mathbf{u}_2 + r\dot{\theta}\frac{d\mathbf{u}_2}{d\theta}\dot{\theta}$$

$$= \ddot{r}\mathbf{u}_1 + \dot{r}\mathbf{u}_2\dot{\theta} + \dot{r}\dot{\theta}\mathbf{u}_2 + r\ddot{\theta}\mathbf{u}_2 - r\dot{\theta}^2\mathbf{u}_1$$

$$= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_2$$

$$\mathbf{a} = \mathbf{u}_1 + \mathbf{u}_2$$

$$\Rightarrow a_r = \ddot{r} - r\dot{\theta}^2 \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

6.6.37

(a) Show that

$$\frac{1}{\rho}\mathbf{n} = \frac{d\mathbf{u}}{ds} = \frac{\mathbf{u}'}{s'} = \frac{(\mathbf{r}'\cdot\mathbf{r}')\mathbf{r}'' - (\mathbf{r}'\cdot\mathbf{r}'')\mathbf{r}'}{|\mathbf{r}'|^4} = \frac{(\mathbf{r}'\times\mathbf{r}'')\times\mathbf{r}'}{|\mathbf{r}'|^4}$$

Solution

$$\frac{d\hat{\mathbf{u}}}{ds} = \frac{\frac{d\hat{\mathbf{u}}}{dt}}{\frac{ds}{dt}} = \frac{\frac{d\hat{\mathbf{u}}}{dt}}{\sqrt{\vec{r}'\cdot\vec{r}'}}$$

$$\mathbf{u}' = \frac{\vec{r}'}{\sqrt{\vec{r}'\cdot\vec{r}'}}, \qquad \sqrt{s' = \vec{r}'\cdot\vec{r}'}$$

$$\frac{d\hat{\mathbf{u}}}{dt} = \frac{(\vec{\mathbf{r}}' \cdot \vec{\mathbf{r}}')\vec{\mathbf{r}}'' - (\vec{\mathbf{r}}' \cdot \vec{\mathbf{r}}'')\vec{\mathbf{r}}'}{(\vec{\mathbf{r}}' \cdot \vec{\mathbf{r}}')^{\frac{3}{2}}}$$

triple vector product

$$\frac{d\hat{\mathbf{u}}}{ds} = \frac{\frac{d\hat{\mathbf{u}}}{dt}}{\sqrt{\vec{r}' \cdot \vec{r}'}} = \frac{(\vec{\mathbf{r}}' \cdot \vec{\mathbf{r}}')\vec{\mathbf{r}}'' - (\vec{\mathbf{r}}' \cdot \vec{\mathbf{r}}'')\vec{\mathbf{r}}'}{|\vec{\mathbf{r}}'|^4} = \frac{(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \times \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}'|^4}$$

(b) Use the identity of langrange [Equation (36)] to obtain the result

$$|(\mathbf{r}' \times \mathbf{r}'') \times \mathbf{r}'| = |\mathbf{r}' \times \mathbf{r}''||\mathbf{r}'|.$$

Solution

Identity of Lagrange

$$(\mathbf{a} \times \mathbf{b}) \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2$$

$$|(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \times \vec{\mathbf{r}}'| = \sqrt{\left((\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \times \vec{\mathbf{r}}'\right) \cdot \left((\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \times \vec{\mathbf{r}}'\right)}$$

$$= \sqrt{(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \cdot (\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'')(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}') - \left((\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \cdot \vec{\mathbf{r}}'\right)^2}$$

o because $(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \cdot \vec{\mathbf{r}}' = (\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \cdot \vec{\mathbf{r}}'$

$$\sqrt{(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'')^2} = |\mathbf{r}' \times \mathbf{r}''||\mathbf{r}'|$$

(c) Deduce the following results

$$\frac{1}{\rho} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}, \qquad \mathbf{n} = \frac{(\mathbf{r}' \times \mathbf{r}'') \times \mathbf{r}'}{|\mathbf{r}' \times \mathbf{r}''||\mathbf{r}'|}$$

Solution

From

$$\frac{1}{\rho}\hat{\mathbf{n}} = \frac{(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}'') \times \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}'|^4}$$

(part a)

$$\frac{1}{\rho} = \frac{\left|\left(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''\right) \times \vec{\mathbf{r}}'\right|}{\left|\vec{\mathbf{r}}'\right|^4} = \frac{\left|\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''\right|\left|\vec{\mathbf{r}}'\right|}{\left|\vec{\mathbf{r}}'\right|^4}$$

$$= \frac{\left|\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''\right|}{\left|\vec{\mathbf{r}}'\right|^3}$$

$$\hat{\mathbf{n}} = \rho \frac{\left(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''\right) \times \vec{\mathbf{r}}'}{\left|\vec{\mathbf{r}}'\right|^4} = \frac{|\mathbf{r}'|^3}{\left|\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''\right|} \frac{\left(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''\right) \times \vec{\mathbf{r}}'}{\left|\vec{\mathbf{r}}'\right|^4}$$

$$= \frac{\left(\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''\right) \times \vec{\mathbf{r}}'}{\left|\vec{\mathbf{r}}'\right|\left|\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''\right|}$$

6.7.47

Determine the unit vector normal to the surface $x^3 - xyz + z^3 = 1$ at the point $(1,1,1)$.

Solution

$$\varphi = x^3 - xyz + z^3 - 1$$

$$\hat{\mathbf{n}} = \frac{\vec{\nabla}\varphi}{\left|\vec{\nabla}\varphi\right|} = \frac{|3x^2 - yz|\hat{\mathbf{i}} + (-xz)\hat{\mathbf{j}} + (3z^2 - xy)\hat{\mathbf{k}}}{\left|\vec{\nabla}\varphi\right|}$$

at $(1,1,1)$

$$\vec{\nabla}\varphi = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

$$\left|\vec{\nabla}\varphi\right| = \sqrt{4+1+4} = 3$$

Therefore

$$\hat{\mathbf{n}} = \pm\frac{1}{3}\left(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}\right)$$

6.8.49

Show that $\nabla \cdot (x\mathbf{v}) = x\nabla \cdot \mathbf{v} + \mathbf{i} \cdot \mathbf{v}$ and $\nabla \times (x\mathbf{v}) = x\nabla \times \mathbf{v} + \mathbf{i} \times \mathbf{v}$.

Solution

Use
$$\vec{\nabla} \cdot (\varphi\, \vec{\mathbf{v}}) = \varphi \nabla \cdot \vec{\mathbf{v}} + \vec{\mathbf{v}}\, \varphi \cdot \vec{\mathbf{v}}$$
$$\vec{\nabla} \times (\varphi\, \vec{\mathbf{v}}) = \varphi \nabla \times \vec{\mathbf{v}} + \vec{\mathbf{v}}\, \varphi \times \vec{\mathbf{v}}$$
$$\varphi = x$$

$$\vec{\nabla} x = \hat{i}$$

therefore
$$\vec{\nabla} \cdot (x\, \vec{\mathbf{v}}) = x\, \vec{\nabla} \cdot \vec{\mathbf{v}} + \hat{\mathbf{i}} \cdot \vec{\mathbf{v}}$$
$$\vec{\nabla} \times (x\, \vec{\mathbf{v}}) = x\, \vec{\nabla} \times \vec{\mathbf{v}} + \hat{\mathbf{i}} \times \vec{\mathbf{v}}$$

6.9.56

Show that $\nabla^2 \mathbf{u} = 0$ if $\nabla \times \mathbf{u} = \nabla \phi$ and $\nabla \cdot \mathbf{u} = 0$, where ϕ is a scalar function.

Solution

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u} = \nabla\overset{0}{(\nabla \cdot \mathbf{u})} - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u}$$
$$\boldsymbol{\nabla} \times (\nabla \phi) = -\nabla^2 \mathbf{u}$$
$$= 0$$

$$\nabla^2 \mathbf{u} = 0$$

6.9.57

If $\mathbf{v} = \phi_1 \nabla \phi_2$, prove $\nabla \times \mathbf{v}$ is $\perp$ to $\mathbf{v}$

Solution

$\nabla \times \mathbf{v} = \nabla \times (\phi_1 \nabla \phi_2)$

Write $\mathbf{u} = \nabla \phi_2$:

$$\nabla \times (\phi_1 \mathbf{u}) = \phi_1 \nabla \times \mathbf{u} + \nabla \phi_1 \times \mathbf{u}$$

$$= \phi_1 \underbrace{\nabla \times \nabla \phi_2}_{\text{curl grad } = 0} + \nabla \phi_1 \times \nabla \phi_2$$

i.e. $\nabla \times \mathbf{v} = \nabla \times \phi_1 \nabla \phi_2 = \nabla \phi_1 \times \nabla \phi_2$

$\mathbf{v} \cdot (\nabla \times \mathbf{v}) = \phi_1 \nabla \phi_2 \cdot (\nabla \phi_1 \times \nabla \phi_2)$; but $\nabla \phi_2 \perp \nabla \phi_1 \times \nabla \phi_2$

and so $\mathbf{v} \cdot (\nabla \times \mathbf{v}) = 0$, i.e. $\mathbf{v} \perp (\nabla \times \mathbf{v})$

6.8.51

$u \cdot (\nabla \phi)$

Prove that $(\mathbf{u} \cdot \nabla)\phi =$

Let $\mathbf{u} = u_z \mathbf{i} + u_y \hat{\jmath} + u_z \hat{\mathbf{k}}$

What does the left-hand side give?

$$\mathbf{u} \cdot \nabla = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}$$

$$(\mathbf{u} \cdot \nabla)\phi = u_z \frac{\partial \phi}{\partial z} + u_y \frac{\partial \phi}{\partial y} + u_z \frac{\partial \phi}{\partial z}$$

What does the right-hand-side give?

$$\nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \hat{\jmath} + \frac{\partial \phi}{\partial z} \hat{\mathbf{k}}$$

$$\mathbf{u} \cdot (\nabla \phi) = u_x \frac{\partial \phi}{\partial x} + u_y \frac{\partial \phi}{\partial y} + u_z \frac{\partial \phi}{\partial z}$$

$$(\mathbf{u} \cdot \nabla)\phi = \mathbf{u} \cdot (\nabla \phi)$$