6.3.10

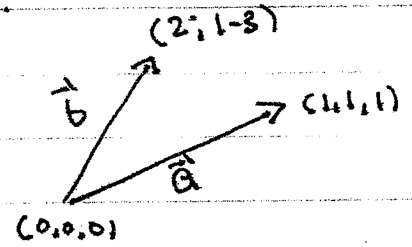
Normal to plane $\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$

Vector $\vec{PP_0}$ in plane : $\vec{PP_0} = (x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}$

Eqn. of plane : $\vec{N}\cdot\vec{PP_0} = (x-x_0)A + (y-y_0)B + (z-z_0)C = 0$

6.3.14.

a)



$\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$

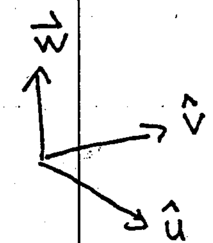
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = \pm(2\hat{i} - \hat{j} - \hat{k})$$ vector normal to plane

Unit vector normal to plane $= \pm(2\hat{i} - \hat{j} - \hat{k})/\sqrt{6}$

b) Area of Triangle $= \frac{1}{2}|\vec{a}\times\vec{b}| = \sqrt{6}/2$

22) $\vec{a} = \alpha_1 \hat{u}$, $\vec{b} = \beta_1 \hat{u} + \beta_2 \hat{v}$,

$\vec{c} = \gamma_1 \hat{u} + \gamma_2 \hat{v} \times \gamma_3 \hat{w}$



Determine λ given,

$$(\vec{a} \times \vec{b}) \times \vec{c} = \lambda \left[(\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a} \right].$$

$$\left[\alpha_1 \hat{u} \times (\beta_1 \hat{u} + \beta_2 \hat{v}) \right] \times (\gamma_1 \hat{u} + \gamma_2 \hat{v} + \gamma_3 \hat{w}) \qquad \rightarrow (\vec{a} \times \vec{b}) \times \vec{c}$$

$$= \alpha_1 \beta_2 (\hat{u} \times \hat{v}) \times (\gamma_1 \hat{u} + \gamma_2 \hat{v} + \gamma_3 \hat{w})$$

$$= \alpha_1 \beta_2 \hat{w} \times (\gamma_1 \hat{u} + \gamma_2 \hat{v} + \gamma_3 \hat{w})$$

$$= \alpha_1 \beta_2 \gamma_1 \hat{w} \times \hat{u} + \alpha_1 \beta_2 \gamma_2 \hat{w} \times \vec{v}$$

$$= \alpha_1 \beta_2 \gamma_1 \hat{v} - \alpha_1 \beta_2 \gamma_2 \hat{u}$$

$$\lambda \left[(\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a} \right] = \lambda \left[\alpha_1 \gamma_1 (\beta_1 \hat{u} + \beta_2 \hat{v}) - (\beta_1 \gamma_1 + \beta_2 \gamma_2) \alpha_1 \hat{u} \right]$$

$$= \lambda \left[\cancel{\alpha_1 \gamma_1 \beta_1 \hat{u}} + \alpha_1 \gamma_1 \beta_2 \hat{v} - \cancel{\alpha_1 \beta_1 \gamma_1 \hat{u}} - \alpha_1 \beta_2 \gamma_2 \hat{u} \right]$$

$$\cancel{= \lambda [\alpha_1 \gamma_1}$$

$$= \lambda \left[\alpha_1 \beta_2 \gamma_1 \hat{v} - \alpha_1 \beta_2 \gamma_2 \hat{u} \right]$$

$$\Rightarrow \boxed{\lambda = 1}$$

6.4.24 Use $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{a})\vec{c}$

$\vec{a} = \hat{i}+\hat{j}+\hat{k} \qquad \vec{b} = \hat{i}+\hat{j} \qquad \vec{c} = \hat{j}+\hat{k}$

$(\hat{i}+\hat{j}+\hat{k})\times\left[(\hat{i}+\hat{j})\times(\hat{j}+\hat{k})\right] = \left[(\hat{i}+\hat{j}+\hat{k})\cdot(\hat{j}+\hat{k})\right](\hat{i}+\hat{j}) - \left[(\hat{i}+\hat{j})\cdot(\hat{i}+\hat{j}+\hat{k})\right]\cdot(\hat{j}+\hat{k}$

$= 2(\hat{i}+\hat{j}) - 2(\hat{j}+\hat{k}) = 2\hat{i} \cancel{+2\hat{j}} \cancel{-2\hat{j}} - 2\hat{k}$

$= 2(i-\hat{k})$

Unit vector in plane of $(\hat{i}+\hat{j})$ and $(\hat{j}+\hat{k}) = \pm \dfrac{(\hat{i}-\hat{k})}{\sqrt{2}}$

6.k.27 (a). Prove that

$$(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d}) = \begin{cases} (\underline{a}\underline{b}\underline{d})\underline{c} - (\underline{a}\underline{b}\underline{c})\underline{d} \\ (\underline{a}\underline{c}\underline{d})\underline{b} - (\underline{b}\underline{c}\underline{d})\underline{a} \end{cases}$$

Let $u = a\times b$. By the identity:

$$\underline{u}\times(\underline{c}\times\underline{d}) = (\underline{u}\cdot\underline{d})\underline{c} - (\underline{c}\cdot\underline{u})\underline{d}$$

$$\Rightarrow (\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d}) = ((\underline{a}\times\underline{b})\cdot\underline{d})\underline{c} - (\underline{c}\cdot(\underline{a}\times\underline{b}))\underline{d}$$

But $(\underline{a}\times\underline{b})\cdot\underline{d} = \underline{a}\cdot(\underline{b}\times\underline{d}) = (\underline{a}\underline{b}\underline{d})$.

Also $\underline{c}\cdot(\underline{a}\times\underline{b}) = (\underline{a}\times\underline{b})\cdot\underline{c} = \underline{a}\cdot(\underline{b}\times\underline{c}) = (\underline{a}\underline{b}\underline{c})$. Thus,

$$(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d}) = (\underline{a}\underline{b}\underline{d})\underline{c} - (\underline{a}\underline{b}\underline{c})\underline{d} \qquad (1)$$

Now let $\underline{u} = \underline{c}\times\underline{d}$. By the identity:

$$(\underline{a}\times\underline{b})\times\underline{u} = (\underline{a}\cdot\underline{u})\underline{b} - (\underline{b}\cdot\underline{u})\underline{a}$$

$$\Rightarrow (\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d}) = (\underline{a}\cdot(\underline{c}\times\underline{d}))\underline{b} - (\underline{b}\cdot(\underline{c}\times\underline{d}))\underline{a}$$

But $\underline{a}\cdot(\underline{c}\times\underline{d}) = (\underline{a}\underline{c}\underline{d})$ and $\underline{b}\cdot(\underline{c}\times\underline{d}) = (\underline{b}\underline{c}\underline{d})$ to give

$$(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d}) = (\underline{a}\underline{c}\underline{d})\underline{b} - (\underline{b}\underline{c}\underline{d})\underline{a} \qquad (2)$$

(b). Show that $(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d})$, if not a zero vector, is in the direction of the intersection of a plane including the vectors $\underline{a}$ and $\underline{b}$ with one including $\underline{c}$ and $\underline{d}$.

Let A_B be a plane containing $\underline{a}$ and $\underline{b}$
" C_D " " " " $\underline{c}$ and $\underline{d}$.

Clearly $(\underline{a}\times\underline{b}) \perp^{ar} A_B$; $(\underline{c}\times\underline{d}) \perp^{ar} C_D$

and $(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{d})$ $\perp^{ar}$ both the normal vectors $(\underline{a} \times \underline{b})$ and $(\underline{c} \times \underline{d})$. Thus $(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{d})$ lies in both planes A_B, C_D. As such, it must be directed along the intersection of planes A_B, C_D (if not zero vector).

This can be checked by taking the scalar product of the vector $(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{d})$ with each of $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ (use (1) and (2)).