Solution Set Ron 11-26-90

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(15) $\nabla^2 T = 0 \qquad T(0,y) = T(\ell,y) = 0\ ; \quad T(x,0) = g(x),\ T(x,d) = F(x)$

$$T = X(x)\,Y(y) \implies X''Y + Y''X = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2 = -\left(\frac{n\pi}{\ell}\right)^2$$

$$* \implies X_n = A_n \sin\left(\frac{n\pi x}{\ell}\right), \quad Y_n = B_n e^{\frac{n\pi}{\ell}y} + C_n e^{-\frac{n\pi}{\ell}y}$$

$$\text{or } = B_n \sinh\left[\frac{n\pi y}{\ell}\right] + C_n \sinh\left[\frac{n\pi(y-d)}{\ell}\right]$$

$$\implies T_n = \left[a_n \sinh\left(\frac{n\pi y}{\ell}\right) + b_n \sinh\left(\frac{n\pi(d-y)}{\ell}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right)$$

Finding constants:

$\underline{y=0}$

$$\sum b_n \sinh\left(\frac{n\pi d}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right) = g(x)$$

$$\implies b_n = \frac{2}{\ell \sinh\left(\frac{n\pi d}{\ell}\right)} \int_0^\ell g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$\underline{y=d}$

$$\sum a_n \sinh\left(\frac{n\pi d}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right) = F(x)$$

$$\implies a_n = \frac{2}{\ell \sinh\left(\frac{n\pi d}{\ell}\right)} \int_0^\ell F(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

(16) a) As above, except $y \to \infty \implies Y_n(y) \to 0$

$\implies B_n$ in $*$ above is $0$ ($B_n = 0$)

$$\implies T = \sum_{n=0}^{\infty} c_n e^{-\frac{n\pi y}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$c_n = \frac{2}{\ell}\int_0^\ell F(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

(16) b) $T(x,y) = \sum_{n=1}^{\infty} \underbrace{a_n}_{\text{from (15)}} \sinh\left(\frac{n\pi(d-y)}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$

$$= \sum_{n=1}^{\infty} c_n \frac{\sinh\left(\frac{n\pi(d-y)}{l}\right)}{\sinh\left(\frac{n\pi d}{l}\right)} \sin\left(n\pi \frac{x}{l}\right)$$

$$= \sum_{n=1}^{\infty} c_n \frac{e^{\frac{n\pi}{l}(d-y)} - e^{-\frac{n\pi}{l}(d-y)}}{e^{\frac{n\pi}{l}d} - e^{-\frac{n\pi}{l}d}} \sin\left(n\pi \frac{x}{l}\right)$$

$$= \sum_{n=1}^{\infty} c_n \frac{e^{-\frac{n\pi}{l}y} - \underbrace{e^{\frac{n\pi}{l}y} e^{-2\frac{n\pi}{l}d}}_{\to 0 \text{ as } d \to \infty}}{1 - \underbrace{e^{-2\frac{n\pi}{l}d}}_{\to 0 \text{ as } d \to \infty}} \sin\left(n\pi \frac{x}{l}\right)$$

$$\Rightarrow \quad T = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi}{l}y} \sin\left(n\pi \frac{x}{l}\right)$$

17 will be posted soon!

57) a) $\partial_t T = -\beta(T - T_0) + \alpha^2 \partial_x^2 T$

$T(x=0,t) = T_1, \quad T(x=\ell, t) = T_2, \quad T(x, t=0) = f(x)$

Let $T = T_0 + U e^{-\beta t}$

$T_t = -\beta U e^{-\beta t} + U_t e^{-\beta t}$

$T_{xx} = U_{xx} e^{-\beta t}$

$\Rightarrow \quad U_t = \alpha^2 U_{xx}, \quad U(x=0,t) = (T_1 - T_0) e^{\beta t}$

$U(x=\ell, t) = (T_2 - T_0) e^{\beta t}$

$U(x, t=0) = f(x) - T_0$

b) $T_1 = T_0 = T_2 \Rightarrow$ Homogeneous BC on $U$

Eqn (161) pg 463 $\Rightarrow U = + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{\ell^2} t} \sin\left(\frac{n\pi x}{\ell}\right)$

(57) b) (cont)

$$a_n = \frac{2}{\ell}\int_0^\ell \left(F(x) - T_0\right) \sin\left(\frac{n\pi x}{\ell}\right) dx = \frac{2}{\ell}\int_0^\ell \left[F - T_0\right] \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$T = T_0 + e^{-\beta t} U$$

$$= T_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n^2\pi^2}{\ell^2}+\beta\right)t} \sin\left(\frac{n\pi x}{\ell}\right)$$

(58) a) $\partial_x^2 T_s = 0 \Rightarrow T_s = a + bx$

$x = 0 \Rightarrow T = T_1 \Rightarrow a = T_1$

$x = \ell \Rightarrow h\ell T_x + (T - T_0) = 0 = h\ell b + a + b\ell - T_0$

$$b\ell(1+h) = T_0 - a = T_0 - T_1$$

$$b = \frac{T_0 - T_1}{\ell(1+h)}$$

$$\Rightarrow T_s = T_1 + \frac{T_0 - T_1}{\ell(1+h)} x$$

b) $T = T_s + T_T \Rightarrow \partial_t T_T = \alpha^2 \partial_x^2 T_T$

$T_T(x=0) = 0, \quad \left(h\ell\, \partial_x T_T + T\right)\Big|_{x=\ell} = 0$

Let $T_T = X(x)\,\tau(t)$

$$X\tau' = \alpha^2 \tau X'' \Rightarrow \frac{X''}{X} = \frac{\tau'}{\alpha^2\tau} = -k^2$$

$X'' + k^2 X = 0, \qquad X(0) = 0 \Rightarrow X = \sin(kx)$

$(h\ell X' + X)_{x=\ell} = 0 \Rightarrow h\ell k\cos(k\ell) + \sin(k\ell) = 0$

$$\tan(k\ell) + h\ell k = 0$$

$k = \frac{k_n}{\ell}$, where $\tan(k_n) + hk_n = 0$

LW

(58) c) $$T_T = \sum a_n \sin\left(\frac{k_n}{\ell} x\right) e^{-\frac{\kappa k_n^2}{\ell^2} t}$$

$$T_T(x, t=0) = f(x) - T_s(x)$$
$$= f(x) - T_1 - \frac{T_2 - T_1}{\ell(1+h)} x$$

$$\Rightarrow \quad a_n \int_0^\ell \sin^2\left(\frac{k_n}{\ell} x\right) dx = \int_0^\ell \left[f(x) - T_s(x)\right] \sin\left(\frac{k_n x}{\ell}\right) dx$$

520 #72

$$w_{h} - c^2 w_{xx} = 0$$

B.C: $w(x,0) = f(x)$ $\quad w_t(x,0) = 0 \quad w(0,t) = 0 = w(l,t)$

a) $w = X(x)T(t) \quad \Rightarrow \quad \dfrac{X''}{x} = \dfrac{T''}{c^2 T} = \lambda$

BC $\Rightarrow \quad \lambda = -\dfrac{n^2\pi^2}{l^2} \qquad X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \qquad (= 0 @ x = 0, l)$

$T(0) \neq 0 \quad T'(0) = 0 \quad \Rightarrow$

$$T = a_n \cos\left(\tfrac{n\pi c t}{l}\right)$$

$$\Rightarrow \quad w = \sum_{n=1}^{\infty} a_n \cos\left(\tfrac{n\pi c t}{l}\right) \sin\left(\tfrac{n\pi x}{l}\right)$$

$$w(x, t=0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\tfrac{n\pi x}{l}\right)$$

Multiply by $\sin\left(\dfrac{n\pi x}{l}\right)$ and $\displaystyle\int^x dx \quad \Rightarrow$

$$a_n \int_0^l \sin^2\left(\tfrac{n\pi x}{l}\right) dx = a_n \frac{l}{2} = \int_0^l f(x) \sin\left(\tfrac{n\pi x}{l}\right) dx$$

b)

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) = \sin(\alpha+\beta) + \sin(\beta-\alpha)$$

$$\Rightarrow \quad w = \sum_{n=1}^{\infty} \frac{a_n}{2}\left[\sin\left(\frac{n\pi}{l}(x-ct)\right) + \sin\left(\frac{n\pi}{l}(x+ct)\right)\right]$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{2}\left[\sin\left(\frac{n\pi}{l}(n-ct)\right) + \sum_{n=1}^{\infty} \frac{a_n}{2} \sin\left(\frac{n\pi}{l}(x+ct)\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{2}F(x-ct)+\frac{1}{2}(F(x+ct)) \\
t &= 0 \quad \Rightarrow \quad F(x)=\sum_{n=1}^{\infty} a_n \sin \sum_{n=1}^{\infty}\left(\frac{n\pi x}{l}\right)
\end{aligned}
$$

Note $F(x)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)=f(x)$ where $f(x)$ is defined $(x\epsilon(0,l))$. Since $F(x)$ is defined by a sum of odd $fcts$, it is odd itself $(F(x)=-F(-x))$. Since $\sin\left(\frac{n\pi x}{l}\right)$ is periodic of period $2l/n, F(x)$ is of period $2l(n=1, \sin\left(\frac{x}{l}\pi\right)$ has period $2l$ which is the largest in the series.)

#96.

$$\bar{\Phi}_{xx} - s^2 \bar{\Phi} + s\varphi(x,0) + \varphi_t(x,0) = \frac{1}{s}$$

$$\bar{\Phi}_{xx} - s^2 \bar{\Phi} = \frac{1}{s} - 1 - s \quad , \quad \bar{\Phi}(0,s) = \frac{1}{s}$$

$$\bar{\Phi}(x,s) = A(s)e^{-sx} + B(s)e^{sx} + K(s)$$

$B = 0$ so $\bar{\Phi}$ is bounded as $x \to \infty$

$$\bar{\Phi}_{xx} - s^2\bar{\Phi} = \cancel{s^2 A(s) e^{-sx}} - s^2\left(\cancel{Ae^{-sx}} + K(s)\right) = 1/s - 1 - s$$

$$\therefore \quad K(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3}$$

Now $\bar{\Phi}(0,s) = A(s) + \cancel{1/s} + 1/s^2 - 1/s^3 = \cancel{\frac{1}{s}}$

$$\therefore \quad A(s) = \frac{1}{s^3} - \frac{1}{s^2}$$

$$\therefore \quad \bar{\Phi}(x,s) = \left(1/s^3 - 1/s^2\right)e^{-sx} + \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3}$$

$$\therefore \quad \varphi(x,t) = \begin{cases} 1 + t - \frac{1}{2}t^2 & t \le x \\ 1 + \cancel{t} - \frac{1}{2}\cancel{t}^2 - (\cancel{t} - x) + \frac{1}{2}(\cancel{t} - x)^2 & \\ = 1 + x - tx + x^2/2 & t \geq x \end{cases}$$