31) $$f(x) = \sum_{n=1}^{\infty} A_n \sin nx \qquad 0 < x < \pi.$$

$$A_n = \frac{\int_0^\pi f(x)\sin(nx)\,dx}{\int_0^\pi \sin^2(nx)\,dx} = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx.$$

NB. $\frac{d}{dx}(\cos nx) = -n\sin nx.$

a) $f(x) = 1$

$$A_n = \frac{2}{\pi}\int_0^\pi \sin(nx)\,dx = -\frac{2}{n\pi}\cos nx\Big|_0^\pi = \boxed{\frac{-2}{n\pi}(\cos n\pi - 1)}$$

$n = 1, 2, 3\ldots$

$$= \frac{2[1-(-1)^n]}{n\pi}$$

$$\therefore \quad 1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin(nx)$$

b) $f(x) = x$

$$A_n = \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx = -\frac{2}{n\pi}\int_0^\pi d(\cos nx).\,x$$

by parts: $$= -\frac{2}{n\pi}\left[x\cos nx\Big|_0^\pi - \int_0^\pi \cos nx\,dx\right]$$

$$= -\frac{2}{n\pi}\left[\pi\cos n\pi\right] - \frac{1}{n}\cdot 0$$

$$= -\frac{2}{n}\cos n\pi = \boxed{(-1)^{n+1}\frac{2}{n}.}$$

$$\therefore \quad x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx).$$

$$\Rightarrow \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

From 31b) : $\quad A_n = (-1)^{n+1} \frac{2}{n}$

$$\therefore \int_0^{\pi} (x)^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} (-1)^{2n+2} \cdot \left(\frac{2}{n}\right)^2$$

$$\rightarrow \quad \frac{x^3}{3}\Big|_0^{\pi} = 2\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\rightarrow \quad \frac{\pi^3}{3} = 2\pi \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)$$

$$\therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots$$

34). $\frac{d^2y}{dx^2} + \lambda y = 0 \; ; \quad y(0) = 0$ ① $\quad l\,y'(L) + k\,y(L) = 0$ ②

c) $$f(x) = \begin{cases} 1 & x < \pi/2 \\ 1/2 & x = \pi/2 \\ 0 & x > \pi/2 \end{cases}$$

$$A_n = \frac{2}{\pi}\left\{ \int_0^{\pi/2} (1).\sin nx\,dx + \int_{\pi/2}^{\pi} (0).\sin nx\,dx \right\}$$

$$= \frac{2}{\pi}\left[\frac{-\cos nx}{n} \right|_0^{\pi/2}$$

$$= \boxed{-\frac{2}{\pi n}\left\{ \cos\frac{n\pi}{2} - 1 \right\}}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \left\{ \frac{1 - \cos\left(\frac{n\pi}{2}\right)}{n} \right\} \sin(nx).$$

34). $$\frac{d^2y}{dx^2} + \lambda y = 0 \; ; \qquad \overset{①}{y(0) = 0} \qquad \overset{②}{l y'(l) + k y(l) = 0,}$$

$k \geq 0.$

Solution: $$y(x) = C \sin\sqrt{\lambda}\, x = C \sin\left(\frac{\mu_n}{l} x\right). \text{ using ①.}$$

$$y'(x) = C\sqrt{\lambda} \cos\sqrt{\lambda}\, x$$

Boundary condition ②:

$$l\, C\sqrt{\lambda} \cos\sqrt{\lambda}\, l + k C \sin\sqrt{\lambda}\, l = 0. \qquad \sqrt{\lambda}\, l = \mu.$$

$$\Rightarrow \; -\tan\sqrt{\lambda}\, l = \frac{l\sqrt{\lambda}}{k}$$

i.e. $$\tan\mu_n = -\frac{\mu_n}{k}$$

$$\mu_n \cos\mu_n + k \sin\mu_n = 0.$$

cont.

$f(x) = 1.$

$$A_n = \frac{\int_0^l (1).\ \sin(\sqrt{\lambda_n}x)\,dx}{\int_0^l \sin^2(\sqrt{\lambda_n}x)\,dx} = \frac{\int_0^l \sin(\sqrt{\lambda_n}x)\,dx}{\frac{1}{2}\int_0^l (1 - \cos(2\sqrt{\lambda_n}x))\,dx}$$

$$= \frac{-\frac{1}{\sqrt{\lambda}}\cos(\sqrt{\lambda_n}x)\Big|_0^l}{\frac{1}{2}\left[x - \frac{1}{2\sqrt{\lambda_n}}\sin(2\sqrt{\lambda_n}x)\right]_0^l} = -\frac{2}{\sqrt{\lambda}} \cdot \frac{(\cos\sqrt{\lambda_n}l - 1)}{\left(l - \frac{1}{2\sqrt{\lambda_n}}\sin(2\sqrt{\lambda_n}l)\right)}$$

$$= \frac{2(1 - \cos(\sqrt{\lambda_n}l))}{(\sqrt{\lambda_n}l) - \frac{1}{2}\sin[2(\sqrt{\lambda_n}l)]}$$

Put $\mu_n = \sqrt{\lambda_n}\,l$:

$$A_n = \frac{2(1 - \cos\mu_n)}{\mu_n - \frac{1}{2}\sin 2\mu_n}$$

but $\sin 2\mu_n = 2\sin\mu_n\cos\mu_n$

$= 2\cos^2\mu_n \cdot \tan\mu_n.$

but $\tan\mu_n = -\frac{\mu_n}{k}$, from before,

$\therefore \sin 2\mu_n = -\frac{2\mu_n}{k}\cos^2\mu_n$, and we have

$$A_n = \frac{2(1 - \cos\mu_n)}{\mu_n\left(1 + \frac{1}{k}\cos^2\mu_n\right)}$$

49)

a) $$f(x) = \begin{cases} 0 & x < 0 \\ x(L-x) & x > 0. \end{cases}$$

<u>Fourier Sine series:</u>

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[L\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx - \int_0^L x^2\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{2}{L}\left[L\left[x\cdot\frac{\cos\left(\frac{n\pi x}{L}\right)}{-\frac{n\pi}{L}}\right|_0^L - L\int_0^L 1.\frac{\cos\left(\frac{n\pi x}{L}\right)}{-\frac{n\pi}{L}}dx\right]$$

$$-\frac{2}{L}\left\{\left.\frac{x^2\cos\left(\frac{n\pi x}{L}\right)}{-\frac{n\pi}{L}}\right|_0^L - \int_0^L 2x.\frac{\cos\left(\frac{n\pi x}{L}\right)}{-\frac{n\pi}{L}}dx\right\}$$

$$= -\frac{2L}{n\pi}\left(L\cancel{\cos(n\pi)}\right) + \frac{2L}{n\pi}\cdot\overset{0}{\cancel{\int_0^L\cos\left(\frac{n\pi x}{L}\right)dx}}$$

cancellation

$$+\frac{2}{n\pi}\left(L^2\cancel{\cos(n\pi)}\right) \quad -\frac{4}{n\pi}\int_0^L x.\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{4}{n\pi}\left\{\overset{0}{\cancel{\left.\frac{x\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right|_0^L}} - \frac{1}{\frac{n\pi}{L}}\int_0^L 1.\sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

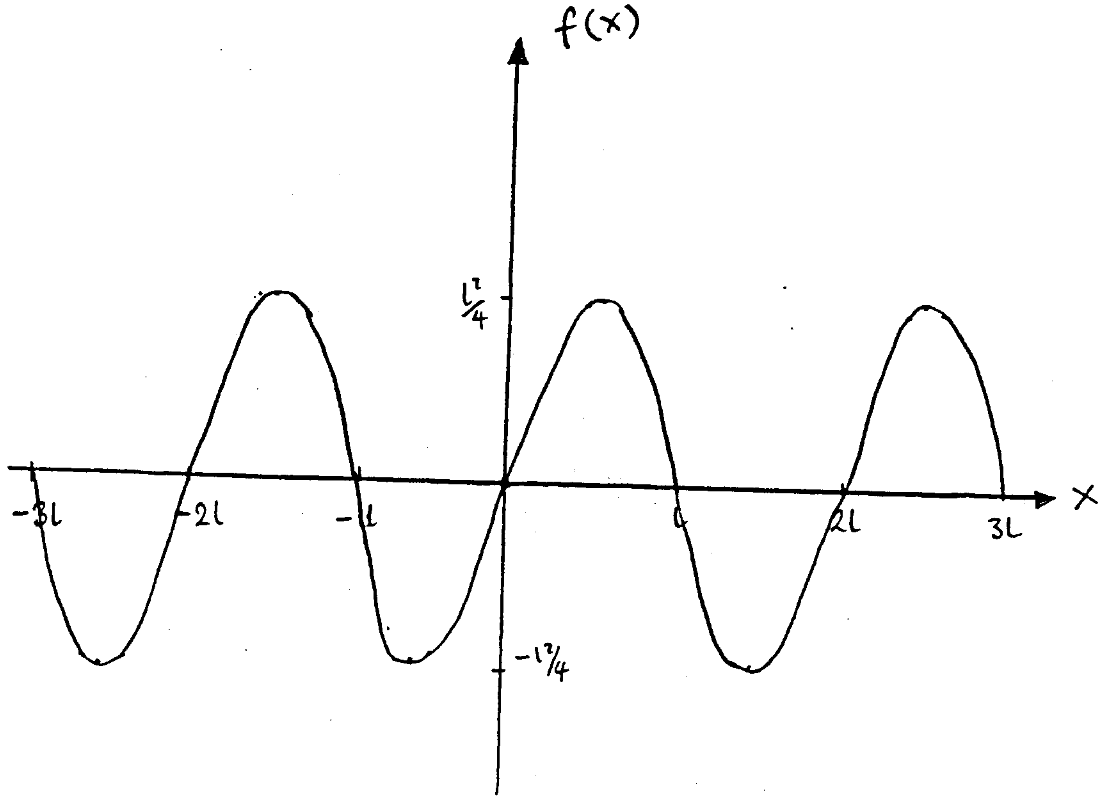
$$= \frac{4L}{(n\pi)^2}\left[\left.\frac{\cos\left(\frac{n\pi x}{L}\right)}{-\left(\frac{n\pi}{L}\right)}\right|_0^L\right.$$

ie $$A_n = \frac{4L^2}{(n\pi)^3}(1-\cos n\pi)$$

$\therefore$ $$A_1 = \frac{8L^2}{1^3\pi^3},\quad A_2 = 0,\quad A_3 = \frac{8L^2}{3^3\pi^3},\ \ldots\ldots$$

$\therefore$ $$f(x) = \frac{8L^2}{\pi^3}\left(\frac{1}{1^3}\sin\frac{\pi x}{L} + \frac{1}{3^3}\sin\frac{3\pi x}{L} + \ldots\ldots\right).$$

) cont.



49 b).

$$f(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < l/2 \\ l-x & x > l/2 \end{cases}$$

$$A_n = \frac{2}{l}\left\{ \int_0^{l/2} x.\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} (l-x)\sin\left(\frac{n\pi x}{l}\right)dx \right\}$$

$$= \frac{2}{l}\left\{ x\,\frac{\cos\left(\frac{n\pi x}{l}\right)}{-\frac{n\pi}{l}}\Bigg|_0^{l/2} - \int_0^{l/2} \frac{\cos\left(\frac{n\pi x}{l}\right)}{-\frac{n\pi}{l}}dx \right.$$

$$\left. + \frac{(l-x)\cos\left(\frac{n\pi x}{l}\right)}{-\frac{n\pi}{l}}\Bigg|_{l/2}^{l} - \int_{l/2}^{l} \frac{(-1)\cos\left(\frac{n\pi x}{l}\right)}{-\left(\frac{n\pi}{l}\right)}dx \right\}$$

$$= \frac{2}{l}\left\{ -\frac{l}{n\pi}\cdot\frac{l}{2}\cos\left(\frac{n\pi}{2}\right) + \frac{l}{n\pi}\,\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\Bigg|_0^{l/2} \right.$$

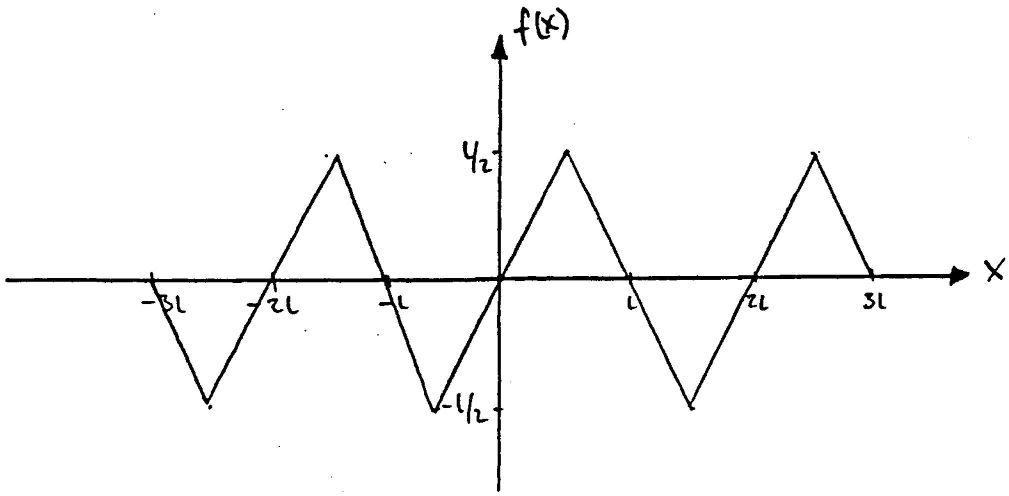
$$\left. + \frac{-l}{n\pi}\cdot\left\{\cancelto{0}{(l-l)}\cos(n\pi) - (l-l/2)\cos\left(\frac{n\pi}{2}\right)\right\} - \frac{l}{n\pi}\,\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\Bigg|_{l/2}^{l} \right\}$$

$$= -\frac{l}{n\pi}\cancel{\cos\left(\frac{n\pi}{2}\right)} + \frac{2l}{(n\pi)^2}\left(\sin\left(\frac{n\pi}{2}\right)\right) + \frac{l}{n\pi}\cancel{\cos\left(\frac{n\pi}{2}\right)} + \frac{2l}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{4l}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore\ A_1 = \frac{4l}{\pi^2 1^2},\quad A_2 = 0,\quad A_3 = \frac{-4l}{\pi^2 3^2},\quad A_4 = 0,\quad A_5 = \frac{4l}{\pi^2 5^2}\ \ldots$$

$$\therefore\ f(x) = \frac{4l}{\pi^2}\left(\frac{1}{1^2}\sin\left(\frac{\pi x}{l}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{l}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi x}{l}\right) - \ldots\right).$$



c).

$$f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left\{\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right)dx - (0)\right\}$$

$$= \frac{2}{L}\left\{ \frac{\cos\frac{n\pi x}{L}}{-\frac{n\pi}{L}}\Bigg|_0^{L/2}\right\}$$

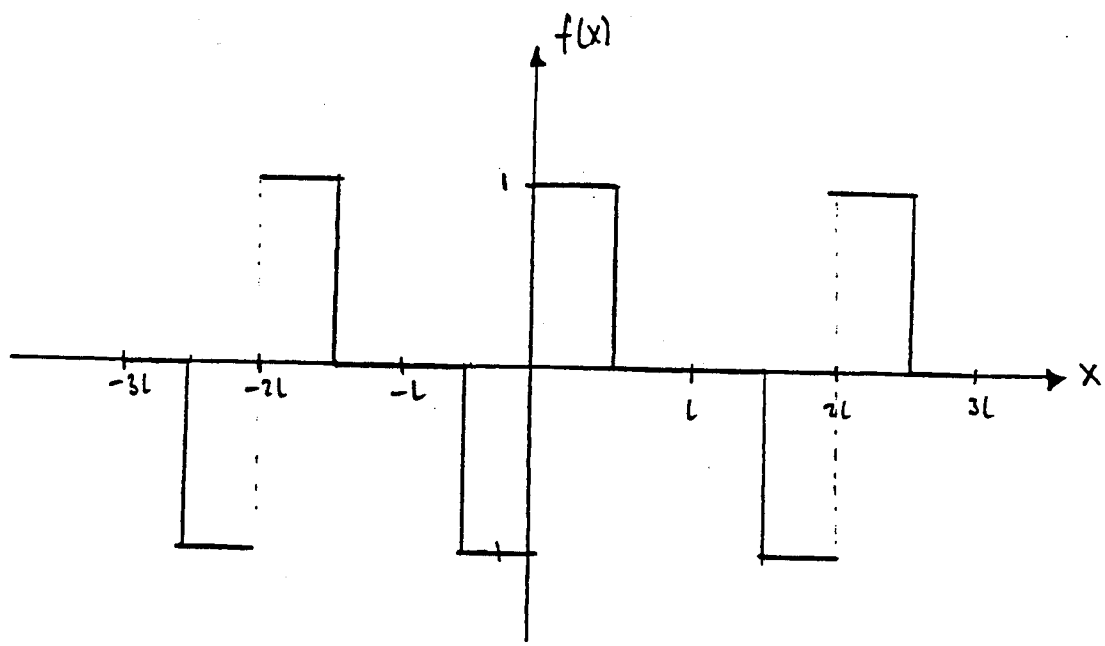
$$= -\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

$$= \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

$A_1 = \frac{2}{\pi}$, $A_2 = \frac{2}{\pi}$, $A_3 = \frac{2}{3\pi}$, ...

$$\therefore f(x) = \frac{2}{\pi}\left(\frac{1}{1}\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \ldots \right.$$

$$\left. \ldots + (1-\cos(\tfrac{n\pi}{2}))\sin\left(\frac{n\pi x}{L}\right) + \ldots\right)$$



49 d). $$f(x) = \sin\left(\frac{\pi x}{2l}\right)$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{2l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

Using: $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$,

we have

$$A_n = \frac{2}{l}\int_0^l \frac{1}{2}\left(\cos\frac{\pi x}{2l}(1-2n) - \cos\frac{\pi x}{2l}(1+2n)\right)dx$$

$$= \frac{1}{l}\left[\frac{\sin\frac{\pi x}{2l}(1-2n)}{\frac{\pi}{2l}(1-2n)} - \frac{\sin\frac{\pi x}{2l}(1+2n)}{\frac{\pi}{2l}(1+2n)}\right|_0^l$$

$$= \frac{2}{\pi}\left(\frac{\sin\frac{\pi}{2}(1-2n)}{(1-2n)} - \frac{\sin\frac{\pi}{2}(1+2n)}{(1+2n)}\right)$$

$$= \frac{2}{\pi}\left(\frac{(1+2n)\left(\sin\frac{\pi}{2}\cos(n\pi) - \cos\frac{\pi}{2}\cancelto{0}{\sin n\pi}\right) - (1-2n)\left(\sin\frac{\pi}{2}\cos n\pi + \cos\frac{\pi}{2}\cancelto{0}{\sin n\pi}\right)}{1-4n^2}\right)$$

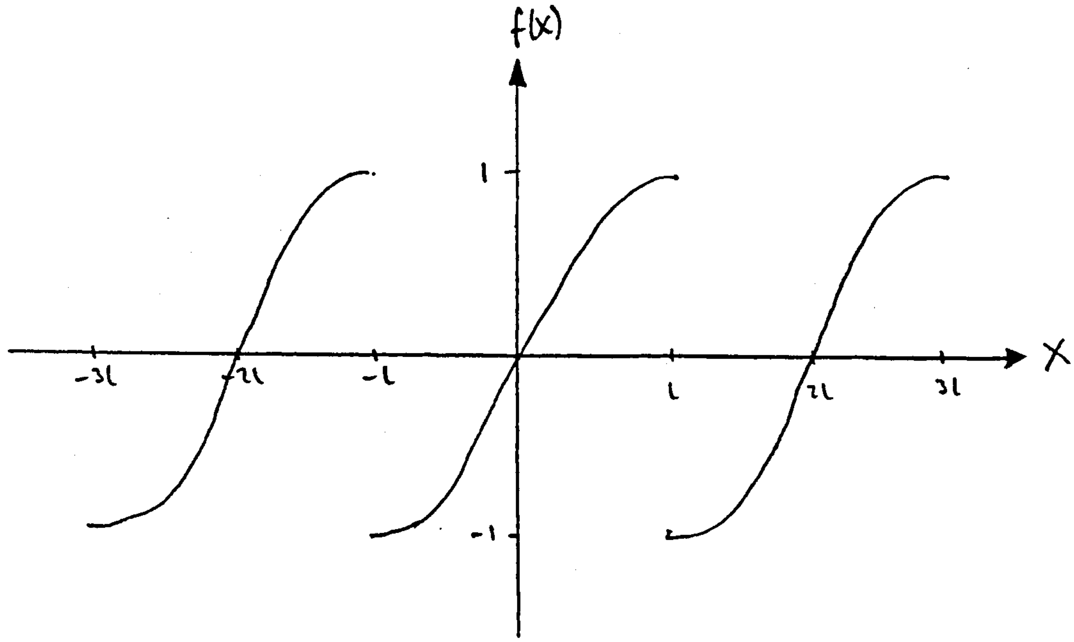
$$= \frac{2}{\pi}\left(\frac{4n\sin\left(\frac{\pi}{2}\right)\cos(n\pi)}{1-4n^2}\right)$$

$$= \frac{8n}{\pi}\,\frac{\cos(n\pi)}{1-4n^2}$$

$$= (-1)^{n+1}\frac{8n}{\pi}\cdot\frac{1}{4n^2-1} = (-1)^{n+1}\frac{8n}{\pi(4n^2-1)}.$$

$$\therefore\ f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{(4n^2-1)}\sin\left(\frac{n\pi x}{l}\right)$$

cont.



49 e) $$f(x) = \begin{cases} \sin\frac{\pi x}{L} & 0 < x < L/2 \\ 0 & \text{otherwise} \end{cases}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^{L/2} \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

use $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right);$

ie $$A_n = \frac{2}{L}\int_0^{L/2}\frac{1}{2}\left(\cos\frac{\pi x}{L}(1-n) - \cos\frac{\pi x}{L}(1+n)\right)dx$$

$$= \frac{1}{L}\left[\frac{\sin\frac{\pi x}{L}(1-n)}{\frac{\pi}{L}(1-n)} - \frac{\sin\frac{\pi x}{L}(1+n)}{\frac{\pi}{L}(1+n)}\right|_0^{L/2}$$

$$= \frac{1}{\pi}\left(\frac{\sin\frac{\pi}{2}(1-n)}{(1-n)} - \frac{\sin\frac{\pi}{2}(1+n)}{(1+n)}\right)$$

$$= \frac{1}{\pi}\left(\frac{\left[\sin\frac{\pi}{2}\cos\frac{n\pi}{2} - \cancelto{0}{\cos\frac{\pi}{2}}\sin\left(-\frac{n\pi}{2}\right)\right](1+n) - (1-n)\left[\sin\frac{\pi}{2}\cos\frac{n\pi}{2} + \cancelto{0}{\cos\frac{\pi}{2}}\sin\frac{n\pi}{2}\right]}{1-n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{2n\cos\left(\frac{n\pi}{2}\right)}{1-n^2}\right)$$

ie $$A_n = \frac{-2n\cos\left(\frac{n\pi}{2}\right)}{\pi(n^2-1)}$$

$$A_1 = \frac{2}{L}\int_0^{L/2}\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_0^{L/2}\left[1-\cos\left(\frac{2\pi x}{L}\right)\right]dx$$

$$= \frac{1}{L}\left[x - \frac{\sin\left(\frac{2\pi x}{L}\right)}{2\pi/L}\right|_0^{L/2}$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}.$$

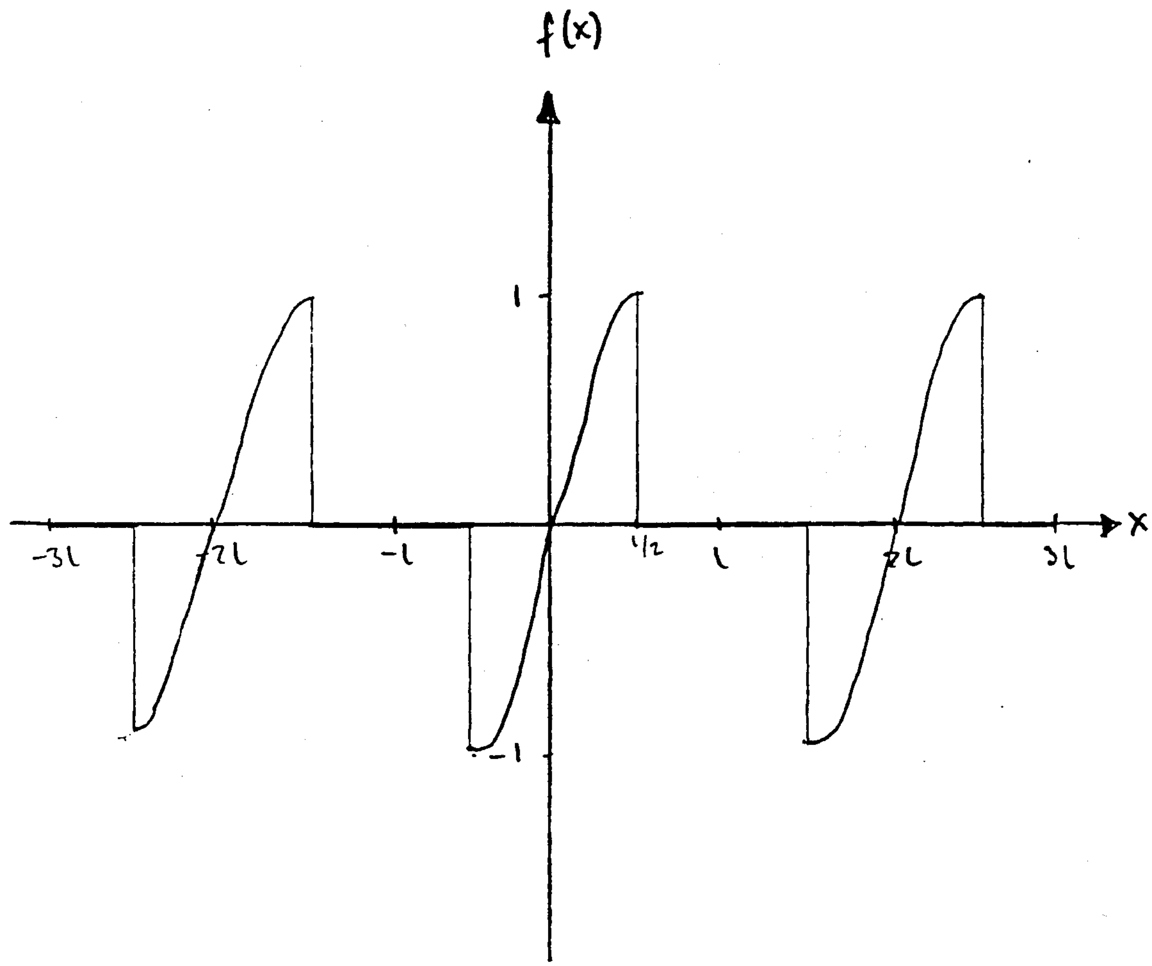
$\therefore$ $$f(x) = \frac{1}{2}\sin\left(\frac{\pi x}{L}\right) - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n\cos\left(\frac{n\pi}{2}\right)}{(n^2-1)}\sin\left(\frac{n\pi x}{L}\right)$$

N.B. n odd $\Rightarrow \cos\frac{n\pi}{2} = 0$

n even $\Rightarrow \cos\frac{n\pi}{2} = \pm 1$

cont.

$$\therefore \quad f(x) = \frac{1}{2}\sin\left(\frac{\pi x}{l}\right) + \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1} m}{4m^2-1} \sin\left(\frac{2m\pi x}{l}\right)$$



49 f) $$f(x) = \begin{cases} \frac{1}{\epsilon} & 0 < x < \epsilon < l \\ 0 & \text{otherwise.} \end{cases}$$

$$A_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

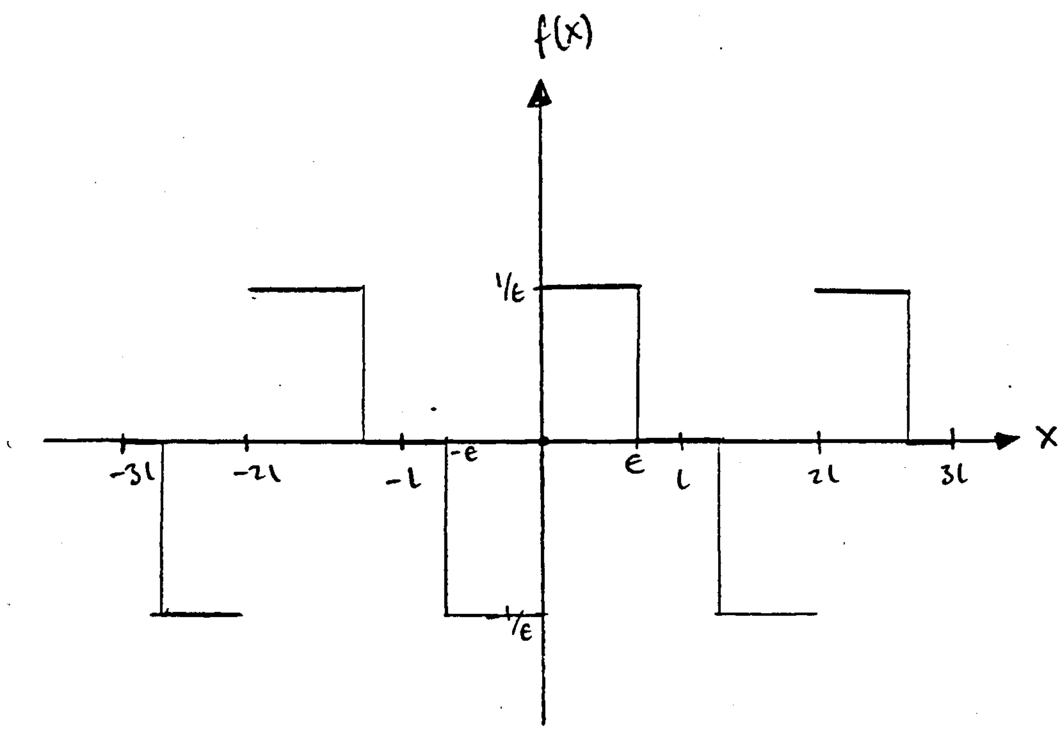
$$= \frac{2}{l}\int_0^\epsilon \frac{1}{\epsilon} \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{\epsilon l}\left[\frac{\cos\left(\frac{n\pi x}{l}\right)}{-\frac{n\pi}{l}} \right|_0^\epsilon$$

$$= \frac{-2}{n\pi\epsilon}\left(\cos\left(\frac{n\pi\epsilon}{l}\right) - 1 \right)$$

ie $$A_n = \frac{2}{\pi}\left[\frac{1 - \cos\left(\frac{n\pi\epsilon}{l}\right)}{n}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 - \cos\left(\frac{n\pi\epsilon}{l}\right)}{n\epsilon} \cdot \sin\left(\frac{n\pi x}{l}\right)$$



a). $f(x) = \cos\alpha x$ is an even function, so use a cosine series.

51 51

a) $$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \qquad 0<x<\pi$$

α nonintegral.

$$= \frac{2}{\pi}\int_0^{\pi}\cos(\alpha x)\cos(nx)\,dx$$

use $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$\therefore \quad A_n = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\{\cos(\alpha+n)x + \cos(\alpha-n)x\}\,dx$$

$$= \frac{1}{\pi}\left\{\frac{\sin(\alpha+n)x}{(\alpha+n)} + \frac{\sin(\alpha-n)x}{(\alpha-n)}\Big|_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left(\frac{\sin(\alpha+n)\pi}{(\alpha+n)} + \frac{\sin(\alpha-n)\pi}{(\alpha-n)}\right)$$

$$= -\frac{1}{\pi}\left(\frac{(\alpha-n)(\sin\alpha\pi\cos n\pi + \cos\alpha\pi\overset{0}{\cancel{\sin n\pi}}) + (\alpha+n)(\sin\alpha\pi\cos n\pi + \overset{0}{\cancel{\sin n\pi}}\cos\alpha\pi)}{n^2-\alpha^2}\right)$$

$$= -\frac{1}{\pi}\,\frac{2\alpha\sin\alpha\pi\cos n\pi}{n^2-\alpha^2}$$

$$\therefore \quad A_n = \frac{2}{\pi}\,\frac{\alpha\sin(\alpha\pi)}{(\alpha^2-n^2)}\cdot(-1)^n$$

$$A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\cos\alpha x\,dx = \frac{1}{\pi}\,\frac{\sin\alpha x}{\alpha}\Big|_0^{\pi}$$

$$= \frac{\sin\alpha\pi}{\alpha\pi}$$

$$\therefore \quad \boxed{\cos(\alpha x) = \frac{\sin\pi\alpha}{\pi\alpha} + \sum_{n=1}^{\infty}(-1)^n\frac{2\alpha\sin(\pi\alpha)}{\pi(\alpha^2-n^2)}\cos(nx)} \quad (-\pi\le x\le\pi)$$

b) $\cot\pi\alpha = \dfrac{\cos\pi\alpha}{\sin\pi\alpha}$, α nonintegral.

Choose $x=\pi$ in part (a):

$$\cos\pi\alpha = \frac{\sin\pi\alpha}{\pi\alpha} + \sum_{n=1}^{\infty}(-1)^n\frac{2\alpha\sin\alpha\pi}{\pi(\alpha^2-n^2)}(-1)^n$$

.51 · b) cont.

$$\therefore \quad \cot \pi\alpha = \frac{\cos\pi\alpha}{\sin\pi\alpha} = \frac{1}{\pi\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\pi(\alpha^2-n^2)}$$

$$= \frac{1}{\pi}\left(\frac{1}{\alpha} - \sum_{n=1}^{\infty} \frac{2\alpha}{(n^2-\alpha^2)}\right)$$

as required.

61. a) $f(x) = A_0 + \sum_{n=1} A_n \cos\left(\frac{n\pi x}{l}\right)$ $\quad 0 \le x \le l$

61

$$[f(x)]^2 = \left[A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)\right]^2$$

$$= \underset{①}{A_0^2} + \underset{②}{2A_0 \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)} + \underset{③}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_n A_m \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)}$$

①: $\frac{2}{l}\int_0^l A_0^2\,dx = 2A_0^2$

②: $\frac{2}{l}\int_0^l 2A_0 \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) dx$ assume we can interchange the order of integration and summation

$$= \frac{4A_0}{l}\sum_{n=1}^{\infty} A_n \underbrace{\int_0^l \cos\left(\frac{n\pi x}{l}\right) dx}_{0}$$

$= 0.$

③: $\frac{2}{l}\int_0^l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_n A_m \cos\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right) dx$ assume we can interchange order of integration and summation

$$= \frac{2}{l}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_n A_m \underbrace{\int_0^l \cos\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right) dx}_{\frac{1}{2} l\,\delta_{mn}}$$

where $\delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m \ne n \end{cases}$

since cosines are orthogonal functions

so only the $m=n$ terms survive, and the double sum reduces to a single sum:

$$= \sum_{n=1}^{\infty} A_n^2$$

$$\therefore \quad \frac{1}{l}\int_0^l [f(x)]^2\,dx = 2A_0^2 + \sum_{n=1}^{\infty} A_n^2.$$

61 b). $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \qquad 0 < x < l$

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \frac{2}{l}\int_0^l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_n B_m \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right) dx$$

assuming we may interchange the order of integration and summation:

$$= \frac{2}{l}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_n B_m \underbrace{\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right) dx}_{\frac{1}{2} l\, \delta_{mn}}$$

$$\text{where} \quad \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

so only the $m=n$ terms survive, and the double reduces to a single sum:

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \sum_{n=1}^{\infty} B_n^2 .$$

c) $f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right) \qquad -l < x < l$

$$= \underbrace{\left(A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)\right)}_{f_a(x)} + \underbrace{\left(\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\right)}_{f_b(x)}$$

Note $f_a(x)$ is an <u>even</u> function, $f_b(x)$ is an <u>odd</u> function.
We have already evaluated $\frac{2}{l}\int_0^l [f_a(x)]^2 dx$ and $\frac{2}{l}\int_0^l [f_b(x)]^2 dx$ in parts a) and b) respectively. It remains to evaluate the cross term:

$$\frac{1}{l}\int_{-l}^{l} 2 f_a(x) f_b(x)\, dx,$$

since $\frac{1}{l}\int_{-l}^{l} [f(x)]^2 dx = \frac{1}{l}\int_{-l}^{l} [f_a(x) + f_b(x)]^2 dx = \frac{2}{l}\int_0^l \overset{\text{even}}{[f_a(x)]^2} dx + \frac{2}{l}\int_0^l \overset{\text{even}}{[f_b(x)]^2} dx$

$$+ \frac{2}{l}\int_0^l f_a(x) f_b(x) dx$$

Q. cont.

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$$\frac{2}{l}\int_{-l}^{l} f_a(x)\, f_b(x)\, dx = \frac{2}{l}\int_{-l}^{l}\left(A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)\right)\cdot\left(\sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi x}{l}\right)\right) dx$$

Note that the integrand is composed of the product of one even and one odd function. Thus the integrand is an <u>odd</u> function of x. Thus the integral will vanish when we integrate over the interval $-l \rightarrow l$.

Note, no calculation was necessary here!

$$\text{Thus } \frac{1}{l}\int_{-l}^{l}[f(x)]^2 dx = \frac{1}{l}\int_{-l}^{l}[f_a(x)]^2 dx + \frac{1}{l}\int_{-l}^{l}[f_b(x)]^2 dx$$

$$= \frac{2}{l}\int_{0}^{l}[f_a(x)]^2 dx + \frac{2}{l}\int_{0}^{l}[f_b(x)]^2 dx$$

$$= \left(2A_0^2 + \sum_{n=1}^{\infty} A_n^2\right) + \left(\sum_{n=1}^{\infty} B_n^2\right) \quad \text{using a), b).}$$

$$\therefore \quad \frac{1}{l}\int_{-l}^{l}[f(x)]^2 dx = 2A_0^2 + \sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right).$$

as required.