8b) $f(x) = e^{-\alpha x}$ $\alpha > 0$ real.

$$f_s(u) = \int_0^\infty e^{-\alpha x} \sin ux \, dx$$

$$= \frac{1}{2i}\int_0^\infty \left(e^{-\alpha x} e^{iux} - e^{-\alpha x} e^{-iux}\right) dx$$

$$= \frac{1}{2i}\left[\int_0^\infty e^{-(\alpha - iu)x} dx - \int_0^\infty e^{-(\alpha+iu)x} dx\right]$$

$$= \frac{1}{2i}\left[-\frac{e^{-(\alpha-iu)x}}{\alpha - iu}\Big|_0^\infty + \frac{e^{-(\alpha+iu)x}}{\alpha+iu}\Big|_0^\infty\right]$$

$$= \frac{1}{2i}\left[\frac{1}{\alpha - iu} - \frac{1}{\alpha+iu}\right] = \frac{1}{2i}\,\frac{\cancel{\alpha}+iu-\cancel{\alpha}+iu}{\alpha^2+u^2} = \frac{u}{\alpha^2+u^2}$$

$$\therefore \quad e^{-\alpha x} = \frac{2}{\pi}\int_0^\infty \frac{u}{\alpha^2+u^2} \sin ux \, du$$

$$f_c(u) = \int_0^\infty e^{-\alpha x} \cos ux \, dx = \frac{1}{2}\left[\int_0^\infty e^{-(\alpha-iu)x} dx + \int_0^\infty e^{-(\alpha+iu)x} dx\right]$$

$$= \frac{1}{2}\left[-\frac{e^{-(\alpha-iu)x}}{\alpha-iu}\Big|_0^\infty - \frac{e^{-(\alpha+iu)x}}{\alpha+iu}\Big|_0^\infty\right]$$

$$= \frac{1}{2}\left[\frac{1}{\alpha-iu} + \frac{1}{\alpha+iu}\right] = \frac{1}{2}\,\frac{\alpha+\cancel{iu}+\alpha-\cancel{iu}}{\alpha^2+u^2} = \frac{\alpha}{\alpha^2+u^2}$$

$$e^{-\alpha x} = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{\alpha^2+u^2} \cos ux \, du \qquad (\alpha > 0,\ x > 0)$$

b) $f(x) = \int_0^\infty A(u) \sin(ux)\,du$

Compute $A(u) = \frac{2}{\pi} \int_0^\infty f(x) \sin(ux)\,dx.$

$= \frac{2}{\pi} \int_0^\infty \frac{x}{\alpha^2 + x^2} \sin(ux)\,dx$

$= e^{-\alpha u}$ using a).

$B(u) = \frac{2}{\pi} \int_0^\infty f(x) \cos(ux)\,du$

$= \frac{2}{\pi} \int_0^\infty \frac{\alpha}{\alpha^2 + x^2} \cos(ux)\,du$

$= e^{-\alpha u}$ using a).

c) use parts a) & b).

37) $f_s(u) = \int_0^\infty e^{-\alpha x} \sin ux \, dx \qquad \alpha = a + ib$

$$= \frac{1}{2i}\left[\int_0^\infty e^{-(a+ib)x} e^{iux} dx - \int_0^\infty e^{-(a+ib)x} e^{-iux} dx\right]$$

$$= \frac{1}{2i}\left[\int_0^\infty e^{-(a+i(b-u))x} dx - \int_0^\infty e^{-(a+i(b+u))x} dx\right]$$

$$= \frac{1}{2i}\left[\frac{1}{(a+i(b-u))} - \frac{}{(a+i(b+u))}\right]$$

$$= \frac{1}{2i}\left[\frac{a+i(b+u) - (a+i(b-u))}{a^2+u^2-b^2+2iab}\right]$$

$$= \frac{u}{a^2-b^2+u^2+2iab}$$

$$\therefore \; e^{-ax}(\cos bx - i\sin bx) = \frac{2}{\pi}\int_0^\infty \frac{u \sin ux}{(a^2-b^2+u^2)+2iab} du$$

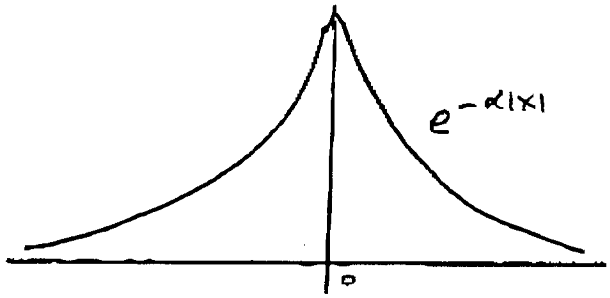
$$= \frac{2}{\pi}\int_0^\infty \frac{[(a^2-b^2+u^2)-2iab]u \sin ux}{(a^2-b^2+u^2)^2+4a^2b^2} dx$$

Equate real and imaginary parts:

$$\therefore \; e^{-ax}\cos bx = \frac{2}{\pi}\int_0^\infty \frac{[a^2-b^2+u^2]u \sin ux \, du}{(a^2-b^2+u^2)^2+4a^2b^2} \qquad (0 < x < \infty)$$

$$\text{d} \quad e^{-ax}\sin bx = \frac{4}{\pi}\int_0^\infty \frac{abu \sin ux}{(a^2-b^2+u^2)^2+4a^2b^2} dx \qquad (0 < x < \infty)$$

88).



$$F(x) = \int_{-\infty}^{\infty} e^{-iux} f(x)\,dx$$

$$= \int_{-\infty}^{0} e^{-iux} e^{\alpha x}\,dx + \int_{0}^{\infty} e^{-iux} e^{-\alpha x}\,dx$$

$$= \int_{-\infty}^{0} e^{(\alpha - iu)x}\,dx + \int_{0}^{\infty} e^{-(\alpha+iu)}\,dx$$

$$= \left.\frac{e^{(\alpha-iu)x}}{\alpha - iu}\right|_{-\infty}^{0} - \left.\frac{e^{-(\alpha+iu)}}{\alpha+iu}\right|_{0}^{\infty}$$

$$= \frac{1}{\alpha - iu} + \frac{1}{\alpha + iu} = \frac{(\alpha+iu) + (\alpha - iu)}{a^2 + u^2}$$

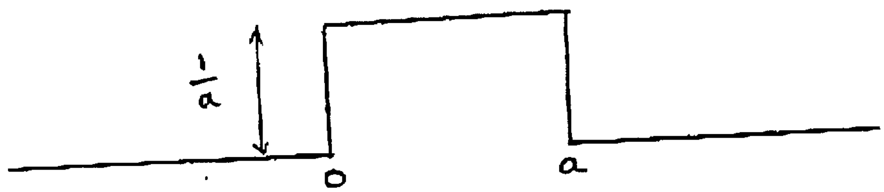
$$= \frac{2\alpha}{\alpha^2 + u^2}.$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{2\alpha}{\alpha^2+u^2}\right) e^{iux}\,du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2+u^2} e^{iux}\,du \qquad (\alpha > 0).$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\pi}\frac{\alpha}{\alpha^2+u^2}\right) e^{iux}\,du \qquad \text{as required.}$$

a)



$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{-iux} f(x)\,dx$$

$$= \int_0^a \frac{e^{-iux}}{a}\,dx$$

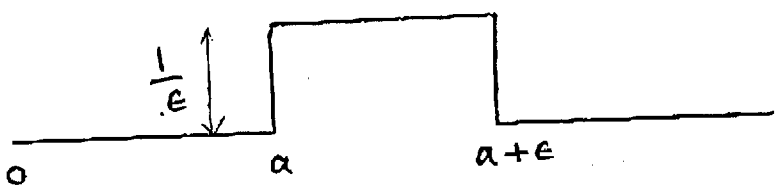
$$= \frac{i\,(e^{-iua}-1)}{au}$$

$$= \frac{1-e^{-iua}}{iau}$$

$$\lim_{a\to 0} \hat{f}(u) = \frac{1-1+iau}{iau} = 1.$$

in the limit $a \to 0$, we have $f(x) \to \delta(x)$.

b)



$$\hat{f}(u) = \int_a^{a+\epsilon} \frac{e^{-iux}}{\epsilon}\,dx = \left.\frac{e^{-iux}}{-iu\epsilon}\right|_a^{a+\epsilon} = \frac{e^{-iu(a+\epsilon)} - e^{-iua}}{-iu\epsilon}$$

$$= e^{-iua}\frac{(e^{-iu\epsilon}-1)}{-iu\epsilon}$$

$$\lim_{\epsilon\to 0} \hat{f}(u) = e^{-iua}\frac{(\cancel{1} - iu\epsilon \ldots - \cancel{1})}{-iu\epsilon}$$

$$= e^{-iau}.$$

9b) Cont

$$S[\delta(x-a)] = \int_0^\infty f(x)\,\text{Sin}\,ux\,dx$$

$$= \int_a^{a+\epsilon} \frac{\text{Sin}\,ux\,dx}{\epsilon} = -\frac{\text{Cos}\,ux}{u\epsilon}\Big|_a^{a+\epsilon}$$

$$= \frac{-\text{Cos}\,u(a+\epsilon) + \text{Cos}\,ua}{u\epsilon}$$

$$= -\frac{\text{Cos}\,ua\,\text{Cos}\,u\epsilon + \text{Sin}\,ua\,\text{Sin}\,u\epsilon + \text{Cos}\,ua}{u\epsilon}$$

$$\lim_{\epsilon \to 0} \frac{-\text{Cos}\,ua(\cancel{1} - u^2\epsilon^2/2 ..) + \text{Sin}\,ua(u\epsilon ..) + \cancel{\text{Cos}}\,ua}{u\epsilon}$$

$$= \text{Sin}\,ua.$$

$$C(\delta(x-a)) = \int_0^\infty f(x)\,\text{Cos}\,ux\,dx$$

$$= \int_a^{a+\epsilon} \frac{\text{Cos}\,ux}{\epsilon}\,dx = \frac{\text{Sin}\,ux}{u\epsilon}\Big|_a^{a+\epsilon}$$

$$= \frac{\text{Sin}\,u(a+\epsilon) - \text{Sin}\,ua}{u\epsilon}$$

$$= \frac{\text{Cos}\,ua\,\text{Sin}\,u\epsilon + \text{Cos}\,u\epsilon\,\text{Sin}\,ua - \text{Sin}\,ua}{u\epsilon}$$

$$\lim_{\epsilon \to 0} = \frac{\text{Cos}(ua)(u\epsilon ..) + (\cancel{1} - \frac{u^2}{2}\epsilon^2 ..)\text{Sin}\,ua - \cancel{\text{Sin}\,ua}}{u\epsilon}$$

$$= \text{Cos}\,au.$$