Heaviside Expansion Formula

Case I If $Q(s)$ contains an unrepeated linear factor $(s-a)$, then $f(t)$ contains the term

$$H(a)\, e^{at}$$

where

$$H(s) = \frac{P(s)\,(s-a)}{Q(s)} \text{ or } H(a) = \frac{P(a)}{Q'(a)}$$

Derivation

We can write

$$\frac{P(s)}{Q(s)} = \frac{A}{s-a} + G(s)$$

in which A is a constant and $G(s)$ has no factor $(s-a)$ in the numerator or denominator. Then

$$f(t) = Ae^{at} + \mathcal{L}^{-1}[G(s)]$$

To solve for A, note that

$$H(s) = \frac{P(s)(s-a)}{Q(s)} = A + (s-a)\,G(s)$$

Then $A = \lim H(s) = H(s) = H(a)$ and the factor $\frac{A}{s-a}$ gives rise to the $s \to a$ term $H(a)\,e^{at}$ in $f(t)$.

Since $Q(a) = 0$, we can also write

$$\frac{P(s)}{Q(s)}(s-a) = P(s)\left(\frac{s-a}{Q(s)-Q(a)}\right) \qquad \text{Letting } s \to a \text{ we get } A = \frac{P(A)}{Q'(a)}$$

where $Q'(a) = \frac{d}{ds}Q(s)\Big|_{s\,=\,a.}$

Case II If $k \geq 2$ and $Q(s)$ contains the linear factor $(s-a)^k$, but not $(s-a)^{k+1}$, then $f(t)$ contains the term

$$\left[\frac{H^{k-1}(a)}{(k-1)!} + \frac{H^{k-2}(a)}{(k-z)!}\cdot\frac{t}{1!} + \frac{H^{(k-3)}(a)}{(k-3)!}\frac{t^2}{2!} + \cdots + \frac{H'(a)}{1!}\frac{t^{k-2}}{(k-2)!} + H(a)\frac{t^{k-1}}{(k-1)!}\right]e^{at}$$

in which

$$H(s) = \frac{P(s)}{Q(s)}(s-a)^k$$

and $H^{(j)}(a)$ denotes the $j-th$ derivative of $H(s)$ evaluated at $s = a$.

Derivation

We can write

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3} + \cdots + \frac{Ak}{(s-a)^k} + G(s)$$

where $(s-a)$ does not appear in $G(s)$.

Then

$$H(s) = \frac{P(s)}{Q(s)}(s-a)^k = A_1(s-a)^{k-1} + A_2(s-a)^{k-2} + \cdots + A_{k-1}(s-a) + A_k$$

$$+ (s-a)^k G(s)$$

$$\lim_{s \to a} H(s) = H(a) = A_k$$

Now compute

$$H''(s) = A_1(k-1)(k-2)(s-a)^{k-3} + A_2(k-2)(k-3)(s-a)^{k-4} + \cdots$$

$$+ 2A_{k-2} + k(k-1)(s-a)^{k-2}G(s) + 2k(s-a)^{k-1}G'(s) + (s-a)^k G''(s)$$

Then $\lim_{s \to a} H''(s) = H''(a) = 2A_{k-2}$ or $A_{k-2} = \frac{1}{2}H''(a)$

Continuing in this fashion, we find that

$$H^{(3)}(a) = (3)(2)A_{k-3}$$

$$H^{(k-1)(a)} = (k-1)(k-2)\cdots(3)(2)A_1$$

Therefore, in general

$$A_{k-j} = \frac{H^{(j)(a)}}{j!} \text{ for } j = 0, 1, 2 \cdots (k-1)$$

Thus,

$$\frac{P(s)}{Q(s)} = \frac{H^{(k-1)(a)}}{(k-1)!} \cdot \frac{1}{s-a} + \frac{H^{(k-2)(a)}}{(k-2)!} \cdot \frac{1}{(s-a)^2} + \frac{H^{(k-3)(a)}}{(k-3)!} \cdot \frac{1}{(s-a)^3} + \cdots$$

$$+ \frac{H^1(a)}{1!} \frac{1}{(s-a)^{k-1}} + H(a) \frac{1}{(s-a)^k} + G(s)$$

Now, we know that

$$\mathcal{L}^{-1}\left[\frac{1}{(s-a)}r\right] = \frac{t^{r-1}}{(r-1)!} e^{at}$$

Thus, we end up with the factor in $f(t)$ corresponding to $(s-a)^k$ in $Q(s)$ is given by

$$\frac{H^{(k-1)(a)}}{(k-1)!} e^{at} + \frac{H^{(k-2)(a)}}{(k-2)!} \frac{t}{1!} e^{at} + \frac{H^{(k-3)(a)}}{(k-3)!} \frac{t^2}{2!} e^{at} + \cdots$$

$$+ \frac{H'(a)}{1!} \frac{t^{k-2}}{(k-2)!} e^{at} + H(a) \frac{t^{k-1}}{(k-1)!} e^{at}$$

as we wanted to show.

Case III If $Q(s)$ contains the unrepeated quadratic factor $(s-a)^2 + b^2$, then $f(t)$ contains the terms

$$\frac{1}{b}\left[\alpha_i \cos(bt) + \alpha_r \sin(bt)\right] e^{at}$$

in which

$$\alpha_r = \mathrm{Re}[H(a+ib)], \quad \alpha_r = \mathrm{Im}[H(a+ib)]$$

and

$$H(s) = \frac{P(s)}{Q(s)}[(s-a)^2 + b^2]$$

<u>Derivation</u>

Write

$$\frac{P(s)}{Q(s)} = \frac{As+B}{(s-a)^2+b^2} + G(s)$$

Where $[(s-a)^2 + b^2]$ does not appear in $G(s)$.

Then

$$H(s) = [(s-a)^2 + b^2]\frac{P(s)}{Q(s)} = As + B + [(s-a)^2 + b^2]\, G(s)$$

Now $\lim_{s \to a+ib} H(s) = H(a+ib) = aA + B + ibA$

Then $\alpha_r = aA + B$ and $\alpha_i = bA$.

Solve for A and B to obtain

$$A = \frac{1}{b}\alpha_i \text{ and } B = \frac{b\alpha_r - a\alpha_i}{b}$$

Then,

$$\frac{P(s)}{Q(s)} = \frac{1}{b}\left[\frac{\alpha_i(s-a)}{(s-a)^2+b^2} + \frac{b\alpha_r}{(s-a)^2+b^2}\right] + G(s)$$

Therefore, the contribution to $f(t)$ from this quadratic factor is

$$\frac{1}{b}\mathcal{L}^{-1}\left[\frac{\alpha_i(s-a)}{(s-a)^2+b^2}\right] + \frac{1}{b}\mathcal{L}^{-1}\left[\frac{b\alpha_r}{(s-a)^2+b^2}\right] = \frac{1}{b}[\alpha_i \cos(bt) + \alpha_r \sin(bt)]\, e^{at}$$

as desired!