These notes start at the solution for an annular disk with inner radius r_1 and outer radius r_2 and outlines the steps in deriving the limiting cases as $r_1 \to 0$ (solid disk) and $r_2 \to \infty$ (infinite sheet with circular hole in middle)

$$T(r,\theta) = (a_0 + b_0 \log r) + \sum_{n=1}^{\infty} \left[\left(a_n r^n + b_n r^{-n}\right)\cos n\theta + \left(c_n r^n + d_n r^{-n}\right)\sin n\theta \right] \qquad r_1 \le r \le r_2 \quad (10)$$

Explicit b.c.'s

$$f_1(\theta) = (a_0 + b_0 \log r_1) + \sum_{n=1}^{\infty} \left[(a_n r_1^n + b_n r_1^{-n})\cos n\theta + (c_n r_1^n + d_n r_1^{-n})\sin n\theta \right]$$

(11)

$$f_2(\theta) = (a_0 + b_0 \log r_2) + \sum_{n=1}^{\infty} \left[(a_n r_2^n + b_n r_2^{-n})\cos n\theta + (c_n r_2^n + d_n r_2^{-n})\sin n\theta \right]$$

But these are complete FOURIER SERIES representations of $f_1(\theta)$ and $f_2(\theta)$

$$a_0 + b_0 \log r_1 = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\, d\theta \qquad (12)$$

$$a_0 + b_0 \log r_2 = \frac{1}{2\pi}\int_0^{2\pi} f_2(\theta)\, d\theta \qquad (13)$$

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\cos n\theta\, d\theta \qquad (14)$$

$$a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\cos n\theta\, d\theta \qquad (15)$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\sin n\theta\, d\theta \qquad (16)$$

$$c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\sin n\theta\, d\theta \qquad (17)$$

Constants a_n, b_n, c_n, d_n $(n = 0, 1, 2, \ldots.)$ are determined from these relations.

Special Limiting Cases – (a) steady state temperature distribution in a disk (interior of circle $r = r_2$).

Take limit $r_1 \to 0$.

In order that solution remains finite we must take $T = f(\theta)$

$$b_0 = b_n = d_n = 0 \quad (n = 1, 2, \ldots.)$$

$\downarrow$

$\log r_1$ is undefined! r_1^{-n} blows up!

Rewrite A_n for $a_n r_2^n$, C_n for $c_n r_2^n$, $a\ for\ r_2$ and $f(\theta)$ for $f_2(\theta)$

and solve the problem

$$\nabla^2 T = 0 \quad (r \le a) \tag{18}$$

With boundary condition $T(a, \theta) = f(\theta)$ (19)

Solution is of the form

$$T(r, \theta) = A_0 + \sum_{n=1}^{\infty} \left(r/a \right)^n \left(A_n \cos n\theta + C_n \sin n\theta \right). \ (r \le a) \tag{20}$$

where

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \ , \quad A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \ , \quad C_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \tag{21}$$

b) Limit $r_2 \to \infty$ Infinite sheet with circular hole in the middle.

$\Rightarrow \quad b_0 = a_n = c_n = 0$

Set $B_n = r_1^{-n} b_n$, $D_n = r_1^{-n} d_n$, $r_1 = a$ and $f(\theta) = f_1(\theta)$, then the solution to the problem

$$\nabla^2 T = 0 (r \ge a) \ ; T(a, \theta) = f(\theta) \tag{22}$$

is of the form

$$T(r, \theta) = A_0 + \sum_{n=1}^{\infty} \left(\frac{a}{r} \right)^n \left(B_n \cos n\theta + D_n \sin n\theta \right) \quad r \ge a \tag{23}$$

where

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta \qquad (n = 1, 2, 3 \ldots.)$$

$$D_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta \qquad (n = 1, 2, 3 \ldots.)$$

Poissons Integral Formula – substitute expressions for expansion coefficients A_n, B_n in (21) into general solutions for $T(r,\theta)$

$$T(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,\mathrm{d}\varphi + \frac{1}{\pi}\sum_{n=1}^{\infty}(\frac{r}{a})^n\left[\cos n\theta\int_0^{2\pi} f(\varphi)\cos n\varphi d\varphi + \sin n\theta\int_0^{2\pi} f(\varphi)\sin n\varphi d\varphi\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\ (\frac{r}{a})^n(\cos n\theta\cos n\varphi + \sin n\theta\sin n\varphi)\right]d\varphi$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\ (\frac{r}{a})^n\cos n(\varphi - \theta)\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\ (\frac{r}{a})^n\,\mathrm{Re}\left[e^{(n(\varphi-\theta))}\right]\right]$$

Use $\sum_{n=1}^{\infty} Z^n = \frac{Z}{1-Z}$ to sum the series with $Z = (\frac{r}{a})e^{i(\varphi-\theta)}$

$$\mathrm{Re}\left[\sum_{n=1}^{\infty}\left[\left(r/a\right)e^{i(\varphi-\theta)}\right]^{n}=\mathrm{Re}[\frac{r/a\, e^{i(\varphi-\theta)}}{1-r/a\, e^{i(\varphi-\theta)}}]\,(r\leq a)\right]$$

$$=\mathrm{Re}\left[\frac{(r/a)[e^{i(\varphi-\theta)}-(r/a)]}{[1-(r/a)[e^{i(\varphi-\theta)}]][1-(r/a)[e^{-i(\varphi-\theta)}]]}\right]$$

$$=\mathrm{Re}\frac{(r/a)\cos(\varphi-\theta)-(r^2/a^2)}{1-2(r/a)\cos(\varphi-\theta)+(r^2/a^2)}$$

$$\therefore\frac{1}{2}+\sum_{n=1}^{\infty}\ r/a^{n}\cos n(\varphi-\theta)=\frac{1}{2}+\frac{(r/a)\cos(\varphi-\theta)-(r^2/a^2)}{1-2\dfrac{r}{a}\cos(\varphi-\theta)+(r^2/a^2)}$$

$$=\frac{1}{2}\frac{1-\dfrac{2r}{a}\cos(\varphi-\theta)+r^2/a^2+\dfrac{2r}{a}\cos(\varphi-\theta)-2(r^2/a^2)}{1-2\,r/a\cos(\varphi-\theta)+(r^2/a^2)}$$

$$\frac{a^2}{a^2}\times\frac{1}{2}\frac{(1-r^2/a^2)}{1-2\,r/a\cos(\varphi-\theta)+(r^2/a^2)}$$

$$T(r,\theta)=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{a^2-r^2}{a^2-2ar\cos(\varphi-\theta)+r^2}T(a,\varphi)d\varphi\quad\leftarrow f(\theta)=T(a,\theta)$$

This is Poisson's Integral formula and shows that the temperature anywhere in the interior of the disk is determined from the temperature distribution (boundary data) around the disk circumference.