The Fourier Integral Transform Pair

Let us rearrange the integrals in (5)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\lambda x} \underbrace{\left\{ \int_{-\infty}^{\infty} e^{-i\lambda t} f(t)\, dt \right\}}_{F(\lambda)} d\lambda$$

We give the inner integral the symbol $F(\lambda)$ as it is a function of λ. Therefore we can now identify the Fourier integral transform pair as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} F(\lambda)\, d\lambda \qquad -\infty < x < \infty$$

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t} f(t)\, dt \qquad -\infty < \lambda < \infty$$

By analogy with the Laplace Transform, we refer to λ as the ~~tr~~ Fourier transform variable.

The idea here again is that in many instances, if we work in the transform space λ, the problem will be greatly simplified and the final step will involve an "inverse" transform back to real space X.

There are also Fourier Sine and Cosine integral transform pairs which follow immediately from the above integrals. ~~when we specialize to odd or even functions f(x)~~

Fourier Sine Integral Transform Pair

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\,\sin(\lambda x)\,d\lambda \qquad 0 < x < \infty$$

$$F_s(\lambda) = \int_0^\infty f(x)\,\sin(\lambda x)\,dx \qquad 0 < \lambda < \infty$$

– represents $-f(-x)$ when $x<0$ i.e. for odd fcn representation is valid $\forall x$

Fourier Cosine Integral Transform Pair.

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\,\cos(\lambda x)\,d\lambda \qquad 0 < x < \infty$$

$$F_c(\lambda) = \int_0^\infty f(x)\,\cos(\lambda x)\,dx \qquad 0 < \lambda < \infty.$$

– represents $f(-x)$ when $x<0$ i.e. for even fcn. representation is valid $\forall x$.

Note that these integrals converge to $f(x)$ in the semi-infinite interval $0 < x < \infty$ whereas the Fourier integral transform converges to $f(x)$ in the infinite interval.

Of course, the function $f(x)$ must be at least, piecewise differentiable on the appropriate interval and at a point of discontinuity the Fourier integral transform (complete, Sine or Cosine) converges to the mean value of the right and left limits of the function there.

Summary:

If $f(x)$ is piecewise differentiable in every finite interval and if the integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ exists, then the Fourier Integral representation of $f(x)$ is valid everywhere, with the usual rule for jump discontinuities.

Note that $F(\lambda)$ is the analog of the expansion coefficients in the Fourier series representation of $f(x)$. Now however, it is a function of a continuous variable λ rather than carrying a discrete index n.

We have a similar "orthogonality" property for the integral transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x} F(\lambda)\,d\lambda$$

Multiply by $e^{-i\lambda' x}$ and integrate ~~from~~ over x from $-\infty$ to $+\infty$

$$\int_{-\infty}^{\infty} e^{-i\lambda' x} f(x)\, dx = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\lambda-\lambda')x}\, dx\right] F(\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} \delta(\lambda-\lambda')\, F(\lambda)\, d\lambda$$

$$= F(\lambda).$$

In other words if $\lambda \neq \lambda'$ the above integral is ZERO.

Problems:

<u>Hand-in</u> 5.10 49) and 51)
5.11 61) and 62)

Home exercise: 5.10) 50) 52) 53)
5.11) 57), 58) 63)

$$\delta(\lambda-\lambda') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\lambda-\lambda')x}\, dx$$