Propagation of Gaussian-apodized paraxial beams through first-order optical systems via complex coordinate transforms and ray transfer matrices

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We investigate the linear propagation of Gaussian-apodized solutions to the paraxial wave equation in free-space and first-order optical systems. In particular, we present complex coordinate transformations that yield a very general and efficient method to apply a Gaussian apodization (possibly with initial phase curvature) to a solution of the paraxial wave equation. Moreover, we show how this method can be extended from free space to describe propagation behavior through nonimaging first-order optical systems by combining our coordinate transform approach with ray transfer matrix methods. Our framework includes several classes of interesting beams that are important in applications as special cases. Among these are, for example, the Bessel–Gauss and the Airy–Gauss beams, which are of strong interest to researchers and practitioners in various fields.

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1. INTRODUCTION

In this paper we investigate the linear propagation of Gaussian-apodized solutions to the paraxial wave equation in free-space and first-order optical systems. In particular, we present complex coordinate transformations that yield a very general and efficient method to apply a Gaussian apodization to a solution of the paraxial wave equation. Moreover, we show how this method can be extended from free space to propagation through nonimaging first-order optical systems by combining our coordinate transform approach with the ABCD matrix analysis framework. The paper is organized as follows. In the introduction we summarize briefly the differential formulation of diffraction theory and outline our main results and the principal ideas for their derivation. We also put the current work in context with previous studies. In Section 2 we develop a framework that accounts for the diffraction or anomalous dispersion effects arising from a Gaussian apodization applied to a solution of the paraxial wave equation via a set of complex coordinate transforms. We state a rigorous result on the connection of the apodized and nonapodized solutions through these coordinate transforms. Furthermore, we show examples of the effectiveness of our method for some important types of beams. In Section 3 we extend the previous results to a very general class of first-order optical systems. Using a minimal optical decomposition of the ray transfer matrix, we reduce the problem to the propagation through thin lenses and free space only. These simplified systems can then be analyzed using our coordinate transforms and the ABCD laws from ray transfer matrix analysis. We end in Section 4 with a summary and discussion of our results.

A. Nondiffracting Solutions of the Paraxial Wave Equation

A commonly used model for the propagation of paraxial waves in a linear isotropic medium is the paraxial wave equation of the form

$$i\frac{\partial u}{\partial z} + \frac{1}{2k}\nabla_x^2 u = 0, \tag{1}$$

with $k = \frac{\omega}{c}$, which is a differential formulation of diffraction theory equivalent to the approach via diffraction integrals; see for example [1,2]. Here we denote by $(\mathbf{x}, z) = (x_1, \ldots, x_d, z)$ the coordinates of a Cartesian coordinate system in $\mathbb{R}^{d+1}$. Furthermore, we use the shorthand notation

$$\nabla_x^2 = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \tag{2}$$

for the transverse Laplacian. Aside from optics partial differential equations (PDEs) of the same type as Eq. (1) arise for example in quantum mechanics [3], electromagnetics [4], and acoustics [5] and are often times referred to as paraxial (wave or Helmholtz) equation or linear Schrödinger equation. As mentioned above, equations of the form Eq. (1) are used for example in optics to study diffraction effects during beam propagation [1,2]. Furthermore, Eq. (1) admits some interesting analytical solutions like the Gaussian beam [6] and Hermite–Gauss and Laguerre–Gauss beams [7–9]. Of particular interest for various applications are also the so-called nondiffracting solutions. The most prominent examples of this class of beams are at the moment probably Bessel [10,11] and Airy beams [3]. However, these theoretical solutions are not always physically realizable in a laboratory. For example, it is well known that a Bessel beam can be viewed as a conical superposition of plane waves [10,12],

$$\int_0^{2\pi} \exp(i\alpha(x\cos\theta + y\sin\theta))\exp(\mathrm{i}n\theta)\mathrm{d}\theta \exp\left(\frac{-\alpha^2 z}{2k}\right) = 2\pi i^n J_n(\alpha r)\exp(\mathrm{i}n\phi)\exp\left(\frac{-\alpha^2 z}{2k}\right),$$

and as such carries infinite energy; see also Fig. 1. In the previous equation we denote by (x, y, z) and (r, ϕ, z) Cartesian and cylindrical coordinates of $\mathbb{R}^3$, respectively. Since physically realizable waves need to have finite energy, a common method to apodize a theoretical wave with infinite energy is the use of a Gaussian apodization; see, for example, [13]. Mathematically, such an apodization can be introduced by multiplying an initial condition (possibly with infinite energy) with a Gaussian bell-shaped function. Thus, the resulting apodized initial condition is ensured to have finite energy. Physically, a Gaussian apodization arises frequently in optics when a laser beam with a Gaussian intensity profile is used. In the case of Bessel beams, the introduction of a Gaussian apodization leads to finite energy analogs usually referred to as Bessel–Gauss beams [14], and, similarly, for Airy beams one may define corresponding Airy–Gauss beams [15]. As an example, Fig. 1 illustrates the generation of a zeroth-order Bessel–Gauss beam with an axicon lens; see for example [12,16].We return to these important examples when we discuss possible applications of our methods in Sections 2 and 3.

In this paper we will introduce a general framework that provides an efficient method to apply a Gaussian apodization—possibly with initial phase curvature—to any solution of Eq. (1) that admits an analytic continuation. In free space, this can be accomplished through a set of complex coordinate transforms that account for the diffraction effects resulting from the apodization. A previous attempt to derive closed-form expression for Gaussian-apodized beams is the study of *Helmholtz–Gauss waves* by Gutiérrez-Vega and Bandres [13]. In the aforementioned paper, the authors derive an analytic expression for Gaussian-apodized nondiffracting solutions to the Helmholtz equation as a product of the complex amplitude of a Gaussian beam, and a scaled transverse profile. Our approach in the present paper differs in important aspects from the discussion in [13] and generalizes the approach significantly. First, we assume a paraxial approximation and thus consider the paraxial wave equation instead of the Helmholtz equation. Second, we allow the Gaussian apodization in the initial conditions to have nonzero phase curvature; i.e., the real part of the complex beam parameter (CBP) does not vanish. Third, we state our main result for arbitrary transverse dimensions; this allows us to treat the practically important cases of one (spatial or temporal), two (spatial), and even three (e.g., two spatial and one temporal) transverse dimensions. The last case is particularly important in situations where diffractive and dispersive effects need to be accounted for; see for example [17] for a discussion of nondiffracting, nondispersive optical light bullets in anomalously dispersive media and their apodized analogs. Fourth, we present a complete derivation and rigorous statement of the transforms and the implications for the solutions of an important class of initial value problems (IVPs) for Eq. (1). Moreover, aside from the significant improvements and generalizations mentioned above, we generalize our coordinate transform method from free-space to nonimaging paraxial optical systems by combining our coordinate transforms with the ABCD laws for ray transfer matrices; see for example [6,18]. Recently, several authors have derived analytical expressions for special classes of beams propagating through first-order (or ABCD) systems; see [15,19–21]. Usually, the starting point of the derivation is the Collins diffraction integral (see [22,23]), and one proceeds by explicitly integrating the resulting expression. Another interesting approach based on the fact that a Bessel–Gauss beam is a conical superposition of Gauss beams is presented in [24]. In this paper we will provide a more general framework that does not require the explicit evaluation of the diffraction integral but instead combines the ray transfer matrix approach with the complex coordinate transforms for free space based on a minimal optical decomposition (see [25]) of the ray transfer matrix of the paraxial system. Furthermore, the method we develop in this paper is not restricted to real ABCD matrices but can even be applied to matrices with complex elements describing optical systems with spatially inhomogeneous loss or gain. Our method is, however restricted to aligned optical systems.

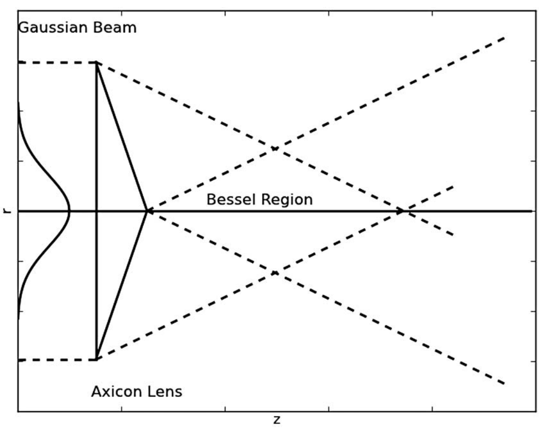

Fig. 1. Creating a conical wave with an axicon lens. The incoming beam on the left is refracted, and the wave vectors after refraction all lie on the surface of a cone. In the diamond-shaped *Bessel region* behind the axicon, the refracted waves form a conical superposition.

B. Gaussian Apodizations in Free-Space and Complex Coordinate Transforms

We start our investigation by considering the Gaussian solutions to Eq. (1) of the form

$$G(\mathbf{x}; z) = \frac{1}{(1 + z/q_0)^{d/2}} \exp\left(\frac{ik\mathbf{x}^2}{2q(z)}\right). \tag{3}$$

The evolution of a radially symmetric Gaussian profile is illustrated in Fig. 2. We assume in Eq. (3) and throughout this

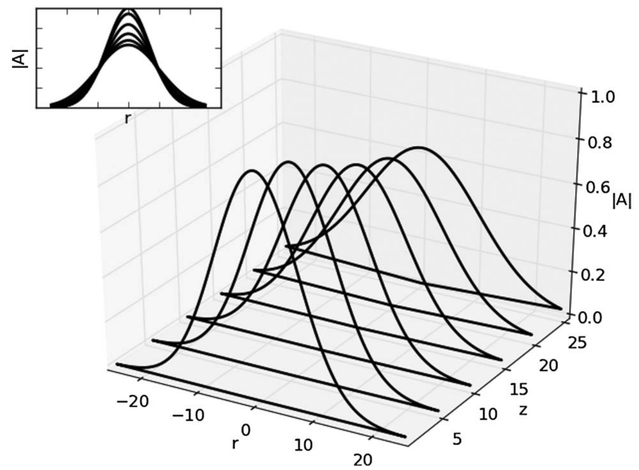

Fig. 2. Evolution of the absolute value of the complex amplitude of a Gaussian profile during propagation in free space. Here we denote $r = \sqrt{x^2 + y^2}$. The numerical values used for this simulation are $k = 0.5$, $w_0 = 10.0$, and $R(0) = 0.0$.

paper that $q_0 \in \mathbb{C}$ and denote by $q(z) = q_0 + z$ the CBP in free space. We describe Gaussian beams and Gaussian apodizations throughout this paper by the CBP ($q(z) = q_0 + z$) since this formulation is very efficient in the ABCD matrix framework. To be reasonably self-contained, we recall here briefly the concept of the CBP in connection to the real spot width and the phase curvature. For a more detailed discussion, we refer for example to Milonni and Eberly [6], Chapter 7. The CBP can be defined by

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{2}{kw^2(z)},$$

where R is the radius of curvature of the phase and w the spot width; see Fig. 3. In this paper we show that, given an analytic continuation U of a solution u of Eq. (1), we obtain a solution corresponding to the apodized initial conditions by the product ansatz

$$v(\mathbf{x}; z) = G(\mathbf{x}; z)U(\tilde{\mathbf{x}}; \tilde{z}), \tag{4}$$

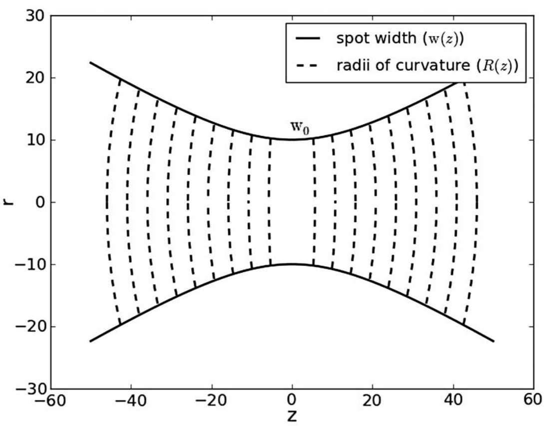

Fig. 3. Illustration of the evolution of spot width and radius of curvature of a Gaussian beam during propagation. Here $w_0 = w(0)$ is called the *beam waist*. The CBP $q(z)$ is defined in terms of the real spot width and radius of curvature by $\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{2}{kw^2(z)}$. The numerical values used for this illustration are $k = 0.5$, $w_0 = 10.0$, and $R(0) = 0.0$.

where, by assumption, $U(\tilde{\mathbf{x}}; \tilde{z})$ is a solution of the paraxial equation

$$i\frac{\partial U}{\partial \tilde{z}} + \frac{1}{2k}\nabla^2_{\tilde{x}} U = 0 \tag{5}$$

in the transformed coordinates

$$\tilde{x}_j(x_j; z) = \frac{x_j}{1 + z/q_0}, \qquad j = 1, 2, \ldots, d, \qquad \tilde{z}(z) = \frac{z}{1 + z/q_0}. \tag{6}$$

In the present paper we derive the above formulas for the coordinate transformation Eqs. (6). Moreover, we can choose the transforms in Eq. (6) such that, if $U(\mathbf{x}; z)$ is the analytic continuation of a solution u of Eq. (1) with initial conditions $u_0(\mathbf{x})$, then $v(\mathbf{x}; z) = G(\mathbf{x}; z)U(\tilde{\mathbf{x}}; \tilde{z})$ is a solution of Eq. (1) with initial conditions $G(\mathbf{x}; 0)u_0(\mathbf{x})$.

C. Propagation in First-Order Optical Systems

Following our discussion in free space, we expand our investigation to the Collins diffraction integral (see [22,23]) and the description of first-order optical systems via *ray transfer or ABCD matrices* (see Fig. 4). In particular, we are interested in finding an analytical expression for the output of a nonimaging first-order optical system in terms of the components of the ray transfer matrix if the input is given by a Gaussian-apodized solution of the paraxial wave equation. For nonimaging first-order optical systems, which by definition satisfy the condition $B \neq 0$, we exploit the fact that the diffraction integral and the corresponding ray transfer matrix of the optical system can be decomposed into expressions corresponding to a *minimal optical decomposition* (see Reference [25]) consisting of a thin lens, free space, and another thin lens (see Fig. 5). Combining our coordinate transforms with the ABCD laws, we derive an analytical expression for the solution of the integral equation

$$v(\mathbf{x}, z_{\text{out}}) = \left(\frac{k}{i2\pi B}\right)^{d/2} \iint_{-\infty}^{\infty} v(\mathbf{x}', z_{\text{in}}) \times \exp\left(\frac{ik}{2B}(A\mathbf{x}'^2 - 2\mathbf{x}' \cdot \mathbf{x} + D\mathbf{x}^2)\right) \mathrm{d}x'_1 \mathrm{d}x'_2$$

of diffraction theory, which is often referred to as *Collins formula*. More precisely, we show in Section 3, Theorem 2

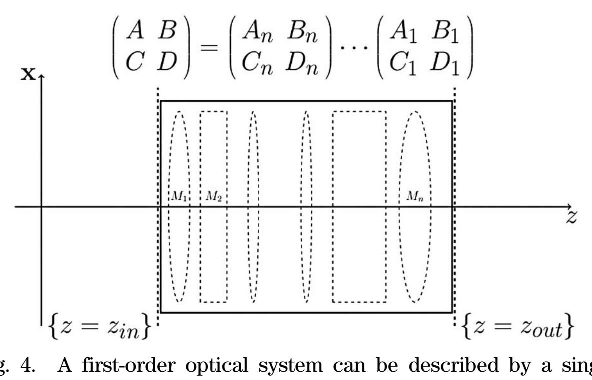

Fig. 4. A first-order optical system can be described by a single ABCD matrix by multiplying the ABCD matrices of the individual components.

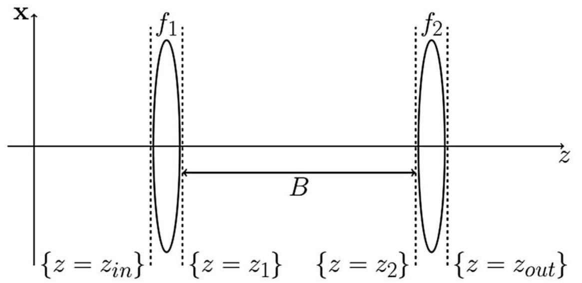

Fig. 5. Propagation through a canonical first-order optical system (ABCD system) consisting of a thin lens L_1 with focal length f_1, free-space propagation over a distance $B \neq 0$, and a second thin lens L_2 with focal length f_2. The B component in the ray transfer matrix corresponds to an effective propagation distance in free space, while the focal distances of the thin lenses are given by $\frac{1}{f_1} = \frac{1-A}{B}$ and $\frac{1}{f_2} = \frac{1-D}{B}$, respectively.

that a solution to the above integral equation with input in the transverse $z = 0$ plane

$$v(\mathbf{x}, 0) = u(\mathbf{x}, 0) \exp\left(\frac{ikx^2}{2q_{\text{in}}}\right),$$

where $u(\mathbf{x}, z)$ is again a solution of the paraxial equation, can be expressed analytically using our complex coordinate transforms. In particular, the output in the $z = z_{\text{out}}$ plane is given by

$$v(\mathbf{x}, z_{\text{out}}) = \frac{1}{\left(A + \frac{B}{q_{\text{in}}}\right)^{d/2}} \exp\left(\frac{ik\mathbf{x}^2}{2q_{\text{out}}}\right) u\left(\frac{\mathbf{x}}{A + B/q_{\text{in}}}, \frac{B}{A + B/q_{\text{in}}}\right),$$

where the CBP is given by

$$\frac{1}{q_{\text{out}}} = \frac{Cq_{\text{in}} + D}{Aq_{\text{in}} + B}.$$

One noteworthy advantage of this method is the fact that it does not require the diffraction integral to be solved explicitly.

2. GAUSSIAN APODIZATION VIA COMPLEX COORDINATES IN FREE SPACE

In this section we derive a set of complex coordinate transforms that provide an elegant method to apply a Gaussian apodization to any solution of the paraxial wave equation Eq. (1) that has an analytic continuation, thus making the apodized solution a solution to the wave equation as well. We start with the derivation followed by a rigorous statement of the main results and some important examples to demonstrate the effectiveness and efficiency of our approach.

A. Derivation of Coordinate Transformations

Assume that we are given a solution $v(\mathbf{x}; z) = G(\mathbf{x}; z)U(\tilde{\mathbf{x}}; \tilde{z})$ as in Eq. (4) of the paraxial equation Eq. (1) where G is again the Gaussian solution defined in Eq. (3). Then inserting the product ansatz for v into the paraxial equation yields the following PDE for U:

$$i\frac{\partial U}{\partial z}(\tilde{\mathbf{x}}; \tilde{z}) + \frac{1}{2k}\nabla^2_{\mathbf{x}} U(\tilde{\mathbf{x}}; \tilde{z}) + \frac{i}{q(z)}\langle \mathbf{x}, \nabla_{\mathbf{x}} U(\tilde{\mathbf{x}}; \tilde{z})\rangle = 0. \quad (7)$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product of Euclidean space. Carrying out the differentiation in Eq. (7), we obtain from the chain rule that $U(\tilde{x}(\mathbf{x}; z); \tilde{z}(\mathbf{x}; z))$ satisfies the equation

$$\begin{aligned} 0 = {} & i\frac{\partial U}{\partial \tilde{z}}\frac{\partial \tilde{z}}{\partial z} + \sum_{k=1}^{2}\frac{\partial^2 U}{\partial \tilde{x}_k^2}\left(\sum_{j=1}^{2}\left(\frac{\partial \tilde{x}_k}{\partial x_j}\right)^2\right) \\ & + \frac{\partial U}{\partial \tilde{z}}\left(\sum_{j=1}^{2}\frac{\partial^2 \tilde{z}}{\partial x_j^2} + \frac{ix_j}{q(z)}\frac{\partial \tilde{z}}{\partial x_j}\right) + \sum_{k=1}^{2}\frac{\partial^2 U}{\partial \tilde{z}\partial \tilde{x}_k}\frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{z}}{\partial x_j} \\ & + \sum_{l=1}^{2}\sum_{\substack{l=1 \\ k\neq 1}}^{2}\frac{\partial^2 U}{\partial \tilde{x}_k \partial \tilde{x}_l}\left(\sum_{j=1}^{2}\frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{x}_l}{\partial x_j}\right) \\ & + \sum_{k=1}^{2}\frac{\partial U}{\partial \tilde{x}_k}\left(i\frac{\partial \tilde{x}_k}{\partial z} + \sum_{j=1}^{2}\left(\frac{\partial^2 \tilde{x}_k}{\partial x_j^2} + \frac{ix_j}{q(z)}\frac{\partial \tilde{x}_k}{\partial x_j}\right)\right). \end{aligned} \quad (8)$$

We observe that the first line of Eq. (8) takes the form of a paraxial equation provided that

$$\frac{\partial \tilde{z}}{\partial z} = \sum_{j=1}^{2}\left(\frac{\partial \tilde{x}_k}{\partial x_j}\right)^2, \qquad k = 1, 2. \quad (9)$$

Furthermore, using Eq. (8), we may impose additional constraints on the coordinate Eqs. (6). In particular, we may assume that the transformation of the propagation coordinate $\tilde{z}$ is independent of the original transverse coordinates x_1, x_2. Moreover, we observe that we may choose the transformations of the transverse coordinates to depend only on one of the original transverse coordinates each. Therefore, we impose that $\frac{\partial \tilde{z}}{\partial x_j} \equiv 0$ for $j = 1$, 2 and $\frac{\partial \tilde{x}_l}{\partial x_j} \equiv 0$ for $j \neq l$. Thus, we may put

$$\frac{\partial \tilde{z}}{\partial z} = \left(\frac{\partial \tilde{x}_j}{\partial x_j}\right)^2 = g^2(z), \qquad j = 1, 2 \quad (10)$$

for some function g of z. Observe that, once we determine g, we have also found

$$\tilde{x}_j = g(z)x_j + c_j(z), \qquad j = 1, 2 \quad (11)$$

for some functions $c_k(z)$ and

$$\tilde{z}(z) = \int g^2(z)\mathrm{d}z. \quad (12)$$

Combining Eq. (11) and the last line on the right-hand side of Eq. (8), we obtain

$$i\frac{dg(z)}{dz}x_j + \frac{dc_j(z)}{dz} + \frac{ix_j}{q(z)}g(z) = 0, \qquad j = 1, 2. \quad (13)$$

Suppose for now that $\frac{dc_j(z)}{dz} \equiv 0$, $j = 1$, 2. Then for $x_j \neq 0$, we conclude from Eq. (13) that

$$\frac{dg(z)}{dz} = -\frac{1}{q_0 + z}g(z), \quad (14)$$

where we used the fact that, in free space, the CBP of a Gaussian beam is given by $q(z) = q_0 + z$. Solving Eq. (14) for $g(0) = 1$, we find the coordinate transforms

$$\tilde{x}_j(x_j; z) = \frac{x_j}{1 + z/q_0} + c_j, \qquad j = 1, 2,$$
$$\tilde{z}(z) = -\frac{{q_0}^2}{(q_0 + z)^2} + c_3.$$

Note that these coordinates are also well defined in the case that $x_j = 0$, which was at first excluded in the derivation. Moreover, if we impose the constraints $\tilde{\mathbf{x}}(\mathbf{x}, 0) = \mathbf{x}$ and $\tilde{z}(\mathbf{x}, 0) = 0$, we obtain the coordinate transforms

$$\tilde{x}_j(x_j; z) = \frac{x_j}{1 + z/q_0}, \qquad j = 1, 2, \qquad \tilde{z}(z) = \frac{z}{1 + z/q_0}.$$

Note that, although we restricted the derivation above to the practically important case of two transverse dimensions ($d = 2$), the same arguments still hold in the more general case ($d \geq 2$), and we obtain the coordinate transforms already stated in Eq. (6).

B. Main Result in Free Space

The previous observations can be summarized in the following statement.

Theorem 1. Let $u:\mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ be the solution of the IVP

$$\begin{cases} -i\frac{\partial u(\mathbf{x};z)}{\partial z} = \frac{1}{2k}\nabla^2_{\mathbf{x}} u(\mathbf{x}; z), \\ u(\mathbf{x}; 0) = u_0(\mathbf{x}). \end{cases}$$

Define for $j = 1, 2, \ldots, d$ the coordinate transforms

$$\tilde{x}_j = \frac{x_j}{1 + z/q_0}, \qquad \tilde{z}(z) = \frac{z}{1 + z/q_0}, \tag{15}$$

and for $q_0 \in \mathbb{C}$, $q(z) = q_0 + z$ denote by

$$G(\mathbf{x}; z) = \frac{1}{(1 + z/q_0)^{d/2}} \exp\left(\frac{ik\mathbf{x}^2}{2q(z)}\right)$$

the solution to the IVP

$$\begin{cases} -i\frac{\partial G(\mathbf{x};z)}{\partial z} = \frac{1}{2k}\nabla^2_{\mathbf{x}} G(\mathbf{x}; z), \\ G(\mathbf{x}; 0) = \exp\left(\frac{ik\mathbf{x}^2}{2q_0}\right). \end{cases}$$

Then the solution of the IVP

$$\begin{cases} -i\frac{\partial v(\mathbf{x};z)}{\partial z} = \frac{1}{2k}\nabla^2_{\mathbf{x}} v(\mathbf{x}; z), \\ v(\mathbf{x}; 0) = \exp\left(\frac{ik\mathbf{x}^2}{2q_0}\right) u_0(\mathbf{x}) \end{cases}$$

is given by

$$v(\mathbf{x}; z) = G(\mathbf{x}; z) U(\tilde{\mathbf{x}}(\mathbf{x}; z); \tilde{z}(\mathbf{x}; z)),$$

where $U: \mathbb{C}^d \times \mathbb{C} \to \mathbb{C}$ is the analytic continuation of u via the complex coordinates $\tilde{\mathbf{x}}$, $\tilde{z}$.

As we can see from Theorem 1, the product structure in the initial conditions is preserved during propagation. For a given paraxial beam, the application of a Gaussian apodization to the initial conditions results in a solution that is the product of the familiar Gaussian solution and the nonapodized beam with complex arguments given by the coordinate transforms. We emphasize here that, in this paper, we allow a more general class of Gaussian apodizations than the methods employed by Gutiérrez-Vega and Bandres in [13] and Mills *et al.* in [17], where the CBP of the initial Gaussian apodization is purely imaginary. This latter situation is contained as a special case in our more general setup. The fact that we have accounted for the possibility of a phase curvature in the Gaussian factor is crucial for the extension of the methods of Theorem 1 from free-space to more general first-order optical systems.

C. Examples

In this section, we show the effectiveness of our method by deriving analytical expressions for Bessel–Gauss beams [10,14]. Furthermore, we discuss classes of Hermite–Gauss and Laguerre–Gauss beams with complex arguments that arise naturally from the complex coordinate transforms in Theorem 1.

1. Bessel and Bessel–Gauss Beams

For $n = 0, 1, 2, \ldots$, a solution of Eq. (1) with initial conditions

$$u(x, y, 0) = 2\pi i^n J_n\left(\alpha\sqrt{x^2 + y^2}\right)\exp(\mathrm{in}\theta)$$
$$= \int_0^{2\pi} \exp(\mathrm{in}\phi)\exp(i\alpha(x\cos(\phi) + y\sin(\phi)))\mathrm{d}\phi$$

is given by the Bessel beam

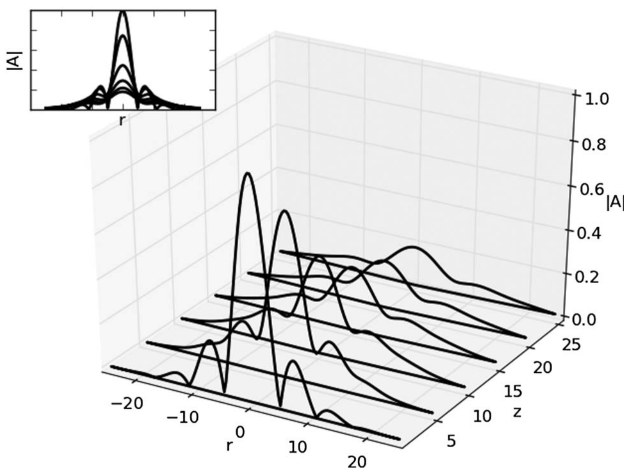

Fig. 6. Evolution of the absolute value of the complex amplitude ($|A|$) of a Bessel–Gauss beam profile during propagation in free space. The propagation characteristics due to dispersion are described by the change of the CBP of the Gaussian factor (broadening of the Gaussian beam width) as well as the complex coordinate transforms (modulation of the Bessel profile). Here we denote $r = \sqrt{x^2 + y^2}$. The numerical values used for this simulation are $\alpha = 0.5$, $k = 0.5$, $w_0 = 10.0$, and $R(0) = 0.0$.

$$u(x,y,z) = 2\pi i^n J_n\left(\alpha\sqrt{x^2+y^2}\right)\exp(\mathrm{i}n\theta)\exp(-i\alpha^2 z/(2k)) = \int_0^{2\pi}\exp(\mathrm{i}n\phi)\exp(i\alpha(x\cos(\phi)+y\sin(\phi)))\mathrm{d}\phi \times\exp(-i\alpha^2 z/(2k)), \quad (16)$$

where $\theta = \tan^{-1}(y/x)$ denotes the azimuthal angle in cylinder coordinates and J_n is the nth-order Bessel function of the first kind. Thus, a Bessel–Gauss beam corresponding to the initial conditions

$$u(x,y,0) = 2\pi i^n \exp(ik(x^2+y^2)/(2q_0))J_n \times\left(\alpha\sqrt{x^2+y^2}\right)\exp(\mathrm{i}n\theta)$$

is given, according to Theorem 1, by

$$u(x,y,z) = 2\pi i^n \frac{1}{1+z/q_0}\exp\left(\frac{ik(x^2+y^2)}{2q(z)}\right)J_n\left(\frac{\alpha\sqrt{x^2+y^2}}{1+z/q_0}\right) \times\exp(\mathrm{i}n\theta)\exp\left(\frac{-i\alpha^2 z}{2k(1+z/q_0)}\right), \quad (17)$$

where $q(z) = q_0 + z$ denotes the CBP in free space. Figure 6 illustrates the evolution of a Bessel–Gauss beam during propagation. We observe that the evolution due to dispersion is characterized by a broadening of the profile described by the change of the CBP of the Gaussian factor as well as a modulation of the Bessel profile due to the complex coordinates.

2. Hermite–Gauss and Laguerre–Gauss Beams with Complex Arguments

In his 1973 paper [7], Siegman introduced a new class of Hermite–Gauss solutions to a paraxial equation of the type Eq. (1). In contrast to the previously considered Hermite–Gauss solutions, these new solutions had the same complex arguments in the Hermite polynomials as in the Gaussian. This idea has been extended to Laguerre–Gauss beams as well and has been studied since by several authors—for example in connection with complex sources [26], perturbation expansions [27,28], and Lie groups and hidden symmetries [8]. We show that our framework applied to the Hermite–Gauss and Laguerre–Gauss solutions with real arguments naturally introduces complex arguments in the Hermite polynomials. Although the form of the coordinate transforms is naturally related to the CBP of the Gaussian beam, it appears as if these apodized versions of Hermite–Gauss and Laguerre–Gauss have not been observed in the literature before.

We start with a Hermite–Gauss solution of the normalized paraxial wave equation

$$i\frac{\partial u(x,y,z)}{\partial z} + \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y,z) = 0,$$

which is given—according to Wünsche [8], Eq. (3.3)—as follows:

$$U(x,y,z) = \frac{w_1 w_2}{(w_1^2+4iz)^{1/2}(w_2^2+4iz)^{1/2}} \times\exp\left(-\frac{x^2}{w_1^2+4iz}-\frac{y^2}{w_2^2+4iz}\right) \times i^m\left(\frac{\alpha_1-4iz}{w_1^2+4iz}\right)^{m/2}H_m\left(x\left(\frac{\alpha_1+w_1^2}{(\alpha_1-4iz)(w_1^2+4iz)}\right)^{1/2}\right) \times i^n\left(\frac{\alpha_2-4iz}{w_2^2+4iz}\right)^{n/2}H_n\left(y\left(\frac{\alpha_2+w_2^2}{(\alpha_2-4iz)(w_2^2+4iz)}\right)^{1/2}\right). \quad (18)$$

In the above, α_j, w_j are positive real numbers for $j = 1, 2$. Furthermore, m and n are positive integers, and H_m denotes the mth Hermite polynomial. Note that, in the case $\alpha_1 = w_1^2$, $\alpha_2 = w_2^2$, we recover the Hermite–Gauss solutions with real arguments. It is a direct consequence of Theorem 1 that, with $k = 1/2$ and $1/q(0) = 4i/w_0^2$, we obtain a new solution:

$$v(x,y,z) = \frac{w_0^2}{w_0^2+4iz}\exp\left(-\frac{x^2+y^2}{w_0^2+4iz}\right)\frac{w_1 w_2}{(w_1^2+4iz)^{1/2}(w_2^2+4iz)^{1/2}}\exp\left(-\frac{x^2}{w_1^2+4iz}-\frac{y^2}{w_2^2+4iz}\right) \times i^m\left(\frac{w_1^2w_0^2+4iz(w_1^2-w_0^2)}{w_1^2w_0^2+4iz(w_1^2+w_0^2)}\right)^{m/2}H_m\left(\frac{w_0^2 x}{w_0^2+4iz}\left(\frac{(\alpha_1+w_1^2)(w_0^2+4iz)^2}{(\alpha_1^2w_0^2+4iz(\alpha_1^2-w_0^2))(w_1^2w_0^2+4iz(w_1^2+w_0^2))}\right)^{1/2}\right) \times i^n\left(\frac{w_2^2w_0^2+4iz(w_2^2-w_0^2)}{w_2^2w_0^2+4iz(w_2^2+w_0^2)}\right)^{n/2}H_n\left(\frac{w_0^2 y}{w_0^2+4iz}\left(\frac{(\alpha_1+w_2^2)(w_0^2+4iz)^2}{(\alpha_2^2w_0^2+4iz(\alpha_2^2-w_0^2))(w_2^2w_0^2+4iz(w_2^2+w_0^2))}\right)^{1/2}\right). \quad (19)$$

In the case $w_0 = w_1 = w_2$, we can collect the first three lines of Eq. (19) into one Gaussian exponential:

$$\frac{w_0^4}{(w_0^2+4iz)^2}\exp\left(-\frac{2(x^2+y^2)}{w_0^2+4iz}\right).$$

But even in the general case, the expression in Eq. (19) can be viewed as a new class of Hermite–Gauss beams. To the best of our knowledge, these solutions form a new class of Hermite–Gauss beams that has not been observed in the literature before.

Similarly to the Hermite–Gauss case, the framework presented above also yields new classes of Laguerre–Gauss solutions if applied to the well-known Laguerre–Gauss solution with real (or possibly complex) arguments that have been studied, for example, in [8,9].

3. PROPAGATION IN NONIMAGING FIRST-ORDER OPTICAL SYSTEMS

In this section we extend our coordinate transform approach from free-space to more general first-order optical systems using some insights from the analysis of ray transfer matrices. More specifically, the main idea consists of factorizing the ray transfer matrix of a general nonimaging system into a minimal optical decomposition (see [25,29]) consisting of two thin lenses and free-space propagation. We can then propagate any Gaussian-apodized solution to the paraxial equation through the system by using our complex coordinate transforms for the free-space section and the well-known *ABCD laws*, e.g., [2,6,18], for the passages through the thin lenses. In this section, unless stated otherwise, we focus on the practically important cases of one and two transverse dimensions $d = 1,2$.

A. Collins Diffraction Integral and Minimal Optical Decompositions

We start by briefly discussing heuristically an observation that motivates the decomposition of the ray transfer matrix into a product corresponding to thin lenses and free-space propagation. Recall that the complex field amplitude $\mathcal{A}_{\text{out}}$ of an electric field immediately after passing through a thin lens in terms of the focal length f and the complex field amplitude immediately before the lens $\mathcal{A}_{\text{in}}$ is given (see for example [30–32]) by

$$\mathcal{A}_{\text{out}}(\mathbf{x}) = \mathcal{A}_{\text{in}}(\mathbf{x}) \exp\left(-ik\frac{\mathbf{x}^2}{2f}\right).$$

The propagation of a paraxial wave through a first-order system (rotationally symmetric if $d = 2$) can be described by a diffraction integral (see [22,23]) where the field $u(\mathbf{x}, z)$ at the output plane $z = z_{\text{out}}$ is given in terms of the field at the input plane $z = z_{\text{in}}$ by

$$u(\mathbf{x}, z_{\text{out}}) = \left(\frac{k}{i2\pi B}\right)^{d/2} \iint_{-\infty}^{\infty} u(\mathbf{x}', z_{\text{in}}) \times \exp\left(\frac{ik}{2B}(A\mathbf{x}'^2 - 2\mathbf{x}'\cdot\mathbf{x} + D\mathbf{x}^2)\right) \mathrm{d}x'_1 \mathrm{d}x'_2. \quad (20)$$

Note that, if $B \neq 0$, we can rewrite Eq. (20) to read

$$u(\mathbf{x}, z_{\text{out}}) = \left(\frac{k}{i2\pi B}\right)^{d/2} \exp\left(\frac{ik}{2B}(D-1)\mathbf{x}^2\right) \iint_{-\infty}^{\infty} u(\mathbf{x}', z_{\text{in}}) \times \exp\left(\frac{ik}{2B}(A-1)\mathbf{x}'^2\right) \times \exp\left(\frac{ik}{2B}(\mathbf{x}'^2 - 2\mathbf{x}'\cdot\mathbf{x} + \mathbf{x}^2)\right) \mathrm{d}x'_1 \mathrm{d}x'_2. \quad (21)$$

The expression under the integral in Eq. (21) can be interpreted as first propagating the input field through a thin lens with focal distance $f_1 = B/(1 - A)$ followed by propagation over a distance B in free space. The exponential that we factored out corresponds to propagation through a second thin lens with focal distance $f_2 = B/(1 - D)$. In the following, we show how this decomposition can be used to solve the integral equation Eq. (20) using our coordinate transforms from Theorem 1. But first we restate the interpretation of the diffraction integral Eq. (21) in terms of thin lenses and free space in the context of decompositions of ray transfer matrices. To do so, we consider a nonimaging first-order optical system, i.e., $B \neq 0$. Then the system is equivalent to a system consisting of a thin lens L_1 with focal length f_1, propagation in free space over the distance B, and a second thin lens L_2 with focal distance f_2; see [25]. Since a thin lens with focal distance f is described by a ray transfer matrix with $A = D = 1$, $C = -1/f$, and $B = 0$, the focal distances f_1 and f_2 are related to the components of the ray transfer matrix of the original system through the relation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix}}_{\text{second lens}} \underbrace{\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}}_{\text{free space}} \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix}}_{\text{first lens}}. \quad (22)$$

Carrying out the matrix multiplication on the right-hand side of Eq. (22) and comparing terms, we recover the reciprocal focal distances given by

$$\frac{1}{f_1} = \frac{1-A}{B}, \qquad \frac{1}{f_2} = \frac{1-D}{B}.$$

In the remainder of this paper, we will refer to a system corresponding to the minimal optical decomposition Eq. (22) of a ray transfer matrix as the associated *canonical optical system*.

With the minimal optical decomposition Eq. (22) at hand, we reduced the task of propagating a Gaussian-apodized paraxial beam through a nonimaging first-order optical system, i.e., solving the integral equation Eq. (20), to the two tasks of propagation through thin lenses and free space. The complex coordinate transforms and Theorem 1 provide an elegant framework for the free-space component, while the effects of a thin lens can be described efficiently using the ABCD laws for the complex amplitude and beam parameter of a Gaussian beam. For completeness we recall that the CBP of a Gaussian q_{out} in a transverse plane immediately behind an ABCD system is given in terms of the new CBP q_{in} in a transverse plane immediately before the system by the formula

$$\frac{1}{q_{\text{out}}} = \frac{C + D/q_{\text{in}}}{A + B/q_{\text{in}}}. \quad (23)$$

Analogously, the amplitude of the Gaussian is modified according to

$$\mathcal{A}_{\text{out}} = \frac{\mathcal{A}_{\text{in}}}{(A + B/q_{\text{in}})^{d/2}}. \quad (24)$$

B. Propagation in Canonical First-Order Optical Systems

We consider a canonical optical system associated with the ray transfer matrix of a first-order optical system for which $B \neq 0$. To propagate the beam through the system using the ABCD frame work, we introduce the planes $P_0 = \{(x, y, z) | z = z_0\}$ and $P_1 = \{(x, y, z) | z = z_1\}$ right in front and behind the first lens. Similarly, we define the planes $P_2 = \{(x, y, z) | z = z_2 = z_1 + B\}$ and $P_3 = \{(x, y, z) | z = z_3\}$ right in front and behind the first lens; see the illustration in Fig. 5.

We start with a Gaussian-apodized solution to the paraxial equation Eq. (1) given by

$$v(\mathbf{x}, z_0) = \frac{1}{\left(1+\frac{z_0}{q_0}\right)^{d/2}} \exp\left(\frac{ik\mathbf{x}^2}{2q_0}\right) u(\mathbf{x}, z_0) \tag{25}$$

immediately before the lens L_1. After passing through the lens, we obtain the modified solution

$$v(\mathbf{x}, z_1) = \frac{1}{\left(1+\frac{z_0}{q_0}\right)^{d/2}} \exp\left(\frac{ik\mathbf{x}^2}{2q_1}\right) u(\mathbf{x}, z_0), \tag{26}$$

where the CBP q_1 is given by the ABCD law for a thin lens as

$$\frac{1}{q_1} = \frac{1}{q_0} - \frac{1}{f_1} = \frac{1}{q_0} + \frac{A-1}{B}. \tag{27}$$

Next, we apply Theorem 1 to propagate the expression in Eq. (26) over a distance B. We obtain at $z_2 = z_1 + B$ the solution

$$v(\mathbf{x}, z_2) = \frac{\left(1+\frac{z_0}{q_1}\right)^{d/2}}{\left(1+\frac{z_0}{q_0}\right)^{d/2}} \frac{1}{\left(1+\frac{z_0+B}{q_1}\right)^{d/2}} \exp\left(\frac{ik\mathbf{x}^2}{2(q_1+B)}\right) \times u\left(\frac{\mathbf{x}}{1+B/q_1}, \frac{z_0+B}{1+B/q_1}\right). \tag{28}$$

Finally, we use the ABCD law again to account for the effects of the lens L_2 and obtain

$$v(\mathbf{x}, z_3) = \frac{\left(1+\frac{z_0}{q_1}\right)^{d/2}}{\left(1+\frac{z_0}{q_0}\right)^{d/2}} \frac{1}{\left(1+\frac{z_0+B}{q_1}\right)^{d/2}} \exp\left(\frac{ik\mathbf{x}^2}{2q_3}\right) \times u\left(\frac{\mathbf{x}}{1+B/q_1}, \frac{z_0+B}{1+B/q_1}\right), \tag{29}$$

where the CBP transforms according to

$$\frac{1}{q_3} = \frac{1}{q_1+B} + \frac{D-1}{B} = \frac{(D-1)}{B} + \frac{(B-q_0+Aq_0)}{(B^2+Aq_0B)} = \frac{Cq_0+D}{Aq_0+B}. \tag{30}$$

We note that an analogous form of Eq. (29) that also allows for misalignment has previously been derived by Bandres and Guizar-Sicairos based on the paraxial group; see Eq. (15) of [33].

C. Main Result

After rewriting $1 + B/q_1 = A + B/q_0$, we obtain the following result from the above observations.

Theorem 2. Let $u(\mathbf{x}, z)$ be a solution of the paraxial equation

$$-i\frac{\partial u}{\partial z} = \frac{1}{2k}\nabla_{\mathbf{x}}^2 u$$

for $d = 1, 2$ transverse dimensions. Suppose that the input to a nonimaging first-order optical system is given in the transverse plane $z = 0$ immediately in front of the system by

$$v(\mathbf{x}, 0) = u(\mathbf{x}, 0)\exp\left(\frac{ikx^2}{2q_{\text{in}}}\right).$$

Then the output in the $z = z_{\text{out}}$ plane immediately behind the optical system is given by

$$v(\mathbf{x}, z_{\text{out}}) = \frac{1}{\left(A+\frac{B}{q_{\text{in}}}\right)^{d/2}} \exp\left(\frac{ik\mathbf{x}^2}{2q_{\text{out}}}\right) \times u\left(\frac{\mathbf{x}}{A+B/q_{\text{in}}}, \frac{B}{A+B/q_{\text{in}}}\right),$$

where the CBP is given by

$$\frac{1}{q_{\text{out}}} = \frac{Cq_{\text{in}}+D}{Aq_{\text{in}}+B},$$

respectively. In particular, v is a solution to the integral equation

$$v(\mathbf{x}, z_{\text{out}}) = \left(\frac{k}{i2\pi B}\right)^{d/2} \int_{\mathbb{R}^d} u(\mathbf{x}', 0) \times \exp\left(\frac{ik}{2B}(A\mathbf{x}'^2 - 2\mathbf{x}'\cdot\mathbf{x} + D\mathbf{x}^2)\right) d\mathbf{x}'.$$

It is noteworthy that, as in Theorem 1, the product structure from the input is preserved during propagation in the case of Theorem 2 as well. For a given paraxial beam, introducing a Gaussian apodization in the initial conditions results in a solution that is the product of the Gaussian solution to the first-order system and the nonapodized beam with complex arguments determined in this case by the coordinate transforms for the free-space propagation distance B and the first thin lens in the canonical optical decomposition. Moreover, it should be emphasized that the above methods and the result in Theorem 2 apply even to complex ABCD matrices.

D. Examples

In this section we derive analytical expressions for some important types of apodized nondiffracting beams such as Bessel–Gauss and Airy–Gauss beams to illustrate the effectiveness of our method.

1. *Airy–Gauss Beams*

We consider the paraxial wave equation Eq. (1) in the case of one transverse dimension $d = 1$ and denote by Ai the well-known Airy function; see for example [34]. Following Bandres and Gutiérrez-Vega (see [15]), we start with the initial conditions

$$u(x, 0) = \exp\left(\frac{S^3 i}{3} + \frac{S(\delta+x)i}{\kappa}\right)\text{Ai}\left(\frac{\delta+x}{\kappa}\right), \tag{31}$$

where $\kappa, \delta, S \in \mathbb{C}$. We can check that a solution is given by an Airy beam of the form

$$u(x,z) = \exp\left(\frac{\left(S+\frac{z}{2\kappa^2 k}\right)^3 i}{3} + \frac{\left(S+\frac{z}{2\kappa^2 k}\right)\left(\delta + x - \frac{z\left(2S+\frac{z}{2\kappa^2 k}\right)}{2\kappa k}\right) i}{\kappa}\right) \times \mathrm{Ai}\left(\frac{\delta+x}{\kappa} - \frac{z\left(2S+\frac{z}{2\kappa^2 k}\right)}{2\kappa^2 k}\right). \quad (32)$$

We return to the case of an Airy–Gauss beam in one transverse dimension and its propagation through a first-order optical system; see [15]. Quite generally we can now determine the analytical form of the Airy–Gauss field after propagation through a nonimaging first-order optical system described by an ABCD matrix with $B \neq 0$ as follows. Applying Theorem 2 to the Airy beam solution in Eq. (32), we obtain immediately the solution

$$u(x,z_{\mathrm{out}}) = \exp\left(\frac{\left(S+\frac{B}{2\kappa^2 k(A+B/q_0)}\right)^3 i}{3}\right) \times \exp\left(\frac{\left(S+\frac{B}{2\kappa^2 k}\right)\left(\delta(A+B/q_0)+x-\frac{B\left(2S+\frac{B}{2\kappa^2 k(A+B/q_0)}\right)}{2\kappa k}\right) i}{\kappa(A+B/q_0)}\right) \times \mathrm{Ai}\left(\frac{\delta(A+B/q_0)+x}{\kappa(A+B/q_0)} - \frac{B\left(2S+\frac{B}{2\kappa^2 k(A+B/q_0)}\right)}{2\kappa^2 k(A+B/q_0)}\right) \times \frac{1}{\sqrt{1+B/q_1}} \exp\left(\frac{ikx^2}{2}\frac{Cq_0+D}{Aq_0+B}\right), \quad (33)$$

which is the Airy–Gauss beam described by Bandres and Gutiérrez-Vega in [15].

2. Bessel–Gauss Beams

We start again with a Bessel beam that is an exact solution to the paraxial equation Eq. (1) in free space. Recall that, for $d = 2$ and $\mathbf{x} = (x, y)$, a Bessel beam is given by Eq. (16). We denote again by J_n the Bessel function of the first kind of integer order n, $\theta = \tan^{-1}(y/x)$ the azimuthal angle in cylinder coordinates, and $q_0 = q(0)$ is the CBP. Thus, for a Bessel–Gauss input

$$v(\mathbf{x},0) = \frac{1}{1+\frac{z_0}{q_0}} \exp\left(\frac{ik(x^2+y^2)}{2q_0}\right) \times J_n\left(\alpha\sqrt{x^2+y^2}\right)\exp(in\theta),$$

immediately before the first lens L_1, it follows from Theorem 2 that, in the $z = z_{\mathrm{out}}$ plane immediately behind the second lens L_2 of the canonical system, the beam is of the form

$$v(\mathbf{x},z_{\mathrm{out}}) = \frac{1}{A+\frac{B}{q_0}} \exp\left(\frac{ik(x^2+y^2)}{2}\frac{Cq_0+D}{Aq_0+B}\right) \times J_n\left(\frac{\alpha\sqrt{x^2+y^2}}{A+\frac{B}{q_0}}\right)\exp(in\theta)\exp\left(-\frac{i\alpha^2 B}{2k\left(A+\frac{B}{q_0}\right)}\right). \quad (34)$$

Note that we derived the expression in Eq. (34) without resolving to explicit integration of the Collins diffraction integral. This latter route was investigated by Belafhal and Dalil-Essakali in [19] and Mei *et al.* in [20]. Unfortunately, the expression derived for a Bessel–Gauss beam in Eq. (12) in [19] seems to be compromised by a typographical error. The more general result for a Bessel–Gauss beam with an annular aperture, which is discussed in [20], reduces to the Bessel–Gauss case as the aperture width increases without bounds. This limiting case is investigated in Eq. (13) in [20], and the derived expression is equivalent to the one in Eq. (34) of the present paper. It is worthwhile to reemphasize at this point the efficiency and elegance of the approach described in this paper. Not only do we avoid the explicit integration of the Collins formula, but the framework developed in Sections 2 and 3 shows clearly the propagation behavior of Gaussian-apodized paraxial wave in terms of the complex amplitude and CBP of a Gaussian beam and the rescaled paraxial wave component with complex arguments. These are important quantities in optics, electromagnetics, and physics and are fundamental for powerful analytical methods such as ray transfer matrices and ABCD laws. Therefore, the expression in Eq. (34) uncovers elegantly the effects of a Gaussian apodization on the propagation behavior of a Bessel–Gauss beam. The combined effects arising from diffraction and a first-order optical system are incorporated in Eq. (34) in terms of ray transfer matrix components as well as the complex amplitude, beam parameter, and coordinate transforms.

4. CONCLUSION

We have developed in Section 2 an efficient method to obtain analytic expressions for Gaussian-apodized analogs of solutions to the paraxial wave equation in free space. In particular, the framework is quite general and can also treat Gaussian apodizations with nonzero phase curvature. This is a crucial detail, which has been exploited in Section 3, where we have used a minimal optical decomposition of nonimaging first-order optical systems into two thin lenses and free space to combine our coordinate transform approach with the ABCD laws from ray transfer matrix analysis. With this approach we have developed a powerful method to compute the output of a nonimaging first-order optical system—provided that the input is a Gaussian-apodized solution of the paraxial wave equation—that avoids the explicit evaluation of the Collins diffraction integral. This method should prove very useful, theoretically and experimentally, when one needs to propagate special solutions of the paraxial wave equation. In particular, our method can efficiently handle, for example, Bessel–Gauss and Airy–Gauss beams. These beams have in recent years attracted the attention of theorists and experimentalists alike, and there is considerable interest in efficient methods to analyze the propagation dynamics of such beams. Aside from an efficient method to obtain analytical expressions for solutions of the paraxial wave equation or the Collins diffraction integral, the results in Theorem 1 and Theorem 2 also reveal an interesting underlying structure of an important class of solutions. It is remarkable that the product structure arising from the initial Gaussian apodization is preserved during propagation and the propagated beam is again given by a product of a Gaussian solution and a scaled version of the nonapodized paraxial solution with complex arguments. This is a striking characteristic of the methods presented in this paper, which is made even more prominent by the fact that they not only preserve and clearly exhibit this product structure but that they also elegantly relate the principal effects arising from diffraction (or possibly anomalous dispersion) and propagation through a nonimaging first-order optical system to the complex amplitude and CBP as well as a complex scaling of the arguments of the paraxial solution.

To expand the possible applications of the methods presented in this paper, we are planning to extend the current framework to include a simultaneous treatment of the temporal and transverse spatial dimensions. Therefore, we expect this work to be of interest not only for linear beams but even in the case of temporally modulated pulses and possibly weakly nonlinear media.

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REFERENCES

1. R. Grella, "Fresnel propagation and diffraction and paraxial wave equation," J. Opt. **13**, 367–374 (1982).
2. T. M. Pritchett and A. D. Trubatch, "A differential formulation of diffraction theory for the undergraduate optics course," Am. J. Phys. **72**, 1026–1034 (2004).
3. M. V. Berry and N. L. Balazs, "Non-spreading wave packets," Am. J. Phys. **47**, 264–267 (1979).
4. M. Levy, *Parabolic Equation Methods for Electromagnetic Wave Propagation*, IEE Electromagnetic Waves Series (Institution of Electrical Engineers, 2000).
5. W. L. Siegmann and D. Lee, "Aspects of three-dimensional parabolic equation computations," Comput. Math. Appl. **11**, 853–862 (1985), Special Issue on Computational Ocean Acoustics.
6. P. W. Milonni and J. H. Eberly, *Laser Physics* (Wiley, 2010).
7. A. E. Siegman, "Hermite–Gaussian functions of complex argument as optical-beam eigenfunctions," J. Opt. Soc. Am. **63**, 1093–1094 (1973).
8. A. Wünsche, "Generalized Gaussian beam solutions of paraxial optics and their connection to a hidden symmetry," J. Opt. Soc. Am. A **6**, 1320–1329 (1989).
9. E. Zauderer, "Complex argument Hermite–Gaussian and Laguerre–Gaussian beams," J. Opt. Soc. Am. A **3**, 465–469 (1986).
10. J. Durnin, "Exact solutions for nondiffracting beams. I. The scalar theory," J. Opt. Soc. Am. A **4**, 651–654 (1987).
11. J. Durnin, J. J. Miceli, and J. H. Eberly, "Diffraction-free beams," Phys. Rev. Lett. **58**, 1499–1501 (1987).
12. D. McGloin and K. Dholakia, "Bessel beams: diffraction in a new light," Contemp. Phys. **46**, 15–28 (2005).
13. J. C. Gutiérrez-Vega and M. A. Bandres, "Helmholtz–Gauss waves," J. Opt. Soc. Am. A **22**, 289–298 (2005).
14. F. Gori, G. Guattari, and C. Padovani, "Bessel–Gauss beams," Opt. Commun. **64**, 491–495 (1987).
15. M. A. Bandres and J. C. Gutiérrez-Vega, "Airy–Gauss beams and their transformation by paraxial optical systems," Opt. Express **15**, 16719–16728 (2007).
16. J. McLeod, "The axicon: a new type of optical element," J. Opt. Soc. Am. **44**, 592–597 (1954).
17. M. S. Mills, G. A. Siviloglou, N. Efremidis, T. Graf, E. M. Wright, J. V. Moloney, and D. N. Christodoulides are preparing a manuscript to be called "Optical bullets with hydrogen-like symmetries."
18. A. E. Siegman. *Lasers* (University Science, 1986).
19. A. Belafhal and L. Dalil-Essakali, "Collins formula and propagation of Bessel-modulated Gaussian light beams through an ABCD optical system," Opt. Commun. **177**, 181–188 (2000).
20. Z. Mei, D. Zhao, X. Wei, F. Jing, and Q. Zhu, "Propagation of Bessel-modulated Gaussian beams through a paraxial ABCD optical system with an annular aperture," Optik **116**, 521–526 (2005).
21. N. Zhou and G. Zeng, "Propagation properties of Hermite–cosine–Gaussian beams through a paraxial optical ABCD system with hard-edge aperture," Opt. Commun. **232**, 49–59 (2004).
22. S. A. Collins, Jr., "Lens-system diffraction integral written in terms of matrix optics," J. Opt. Soc. Am. **60**, 1168–1177 (1970).
23. J. A. Arnaud, *Beam and Fiber Optics* (Academic, 1976), Chap. 2.
24. M. Santarsiero, "Propagation of generalized Bessel–Gauss beams through ABCD optical systems," Opt. Commun. **132**, 1–7 (1996).
25. X. Liu and K.-H. Brenner, "Minimal optical decomposition of ray transfer matrices," Appl. Opt. **47** E88–E98 (2008).
26. S. Y. Shin and L. B. Felsen, "Gaussian beam modes by multipoles with complex source points," J. Opt. Soc. Am. **67**, 699–700 (1977).
27. G. P. Agrawal and D. N. Pattanayak, "Gaussian beam propagation beyond the paraxial approximation," J. Opt. Soc. Am. **69**, 575–578 (1979).
28. T. Takenaka, M. Yokota, and O. Fukumitsu, "Propagation of light beams beyond the paraxial approximation," J. Opt. Soc. Am. A **2**, 826–829 (1985).
29. L. W. Casperson, "Synthesis of Gaussian beam optical systems," Appl. Opt. **20**, 2243–2249 (1981).
30. G. R. Fowles, *Introduction to Modern Optics*, 2nd ed. (Dover, 1989).
31. J. W. Goodman, *Introduction to Fourier Optics*, McGraw-Hill Physical and Quantum Electronics Series (Roberts, 2005).
32. A. Yariv, *Optical Electronics*, The Holt, Rinehart, and Winston Series in Electrical Engineering (Saunders College, 1991).
33. M. A. Bandres and M. Guizar-Sicairos, "Paraxial group," Opt. Lett. **34**, 13–15 (2009).
34. National Institute of Standards and Technology, Digital Library of Mathematical Functions (29 August 2011), http://dlmf.nist.gov/.
35. E. Jones, T. Oliphant, and P. Peterson, "SciPy: open source scientific tools for Python" (2001), http://www.scipy.org.
36. J. D. Hunter, "Matplotlib: a 2D graphics environment," Comput. Sci. Eng. **9**, 90–95 (2007).